THE MAY MEETING OF THE ILLINOIS SECTION.

The fifth annual meeting of the Illinois Section of the Mathematical Association of America was held in the high school building at Elgin, Illinois, on May second and third, 1924, in conjunction with the Illinois State Academy of Science. There were three sessions. On Friday afternoon, Chairman Comstock called the meeting to order at 2:00. The Friday evening session was a joint meeting of the Academy and of the Illinois Section. Professor Comstock presided at the Saturday morning session, beginning at 9:30.

The attendance was thirty, including the following seventeen members of the Association: A. B. Coble, C. E. Comstock, M. W. Coultrap, D. R. Curtiss, C. D. Garlough, Marie M. Johnson, E. P. Lane, Martha P. McGavock, Bessie I. Miller, E. J. Moulton, Mary W. Newson, C. I. Palmer, H. P. Pettit, Marie Plapp, G. H. Scott, G. T. Sellew, F. E. Wood.

The following officers were elected: E. J. MOULTON, Chairman; E. B. LYTLE, Vice-Chairman; BESSIE I. MILLER, Secretary-Treasurer. By a unanimous vote it was decided to leave the time and place of the next meeting of the Section to the decision of a special committee consisting of the executive committee and Professors Slaught and Scott, with the understanding that the meeting will be separate from that of the Academy of Science.

After the reading of the minutes of the 1923 meeting, the following papers were presented:

Friday afternoon.

(1) "College geometry for teachers" by Professor F. E. WOOD, Northwestern University.

(2) "New proofs in the theory of functions" by Professor D. R. CURTISS, Northwestern University.

(3) "Report on algebraic geometry" by Professor A. B. COBLE, University of Illinois.

Saturday morning.

(4) "Courses in mathematics for freshmen" by Miss MARY BEATY, Illinois College.

(5) General discussion on the paper, opened by Professor A. E. GAULT, Bradley Polytechnic Institute.

Abstracts of papers, numbered as in the above list, follow:

1. In this paper it is argued that a course of mathematical type designed for teachers should

(1) be specific, yet have general applications,

(2) deal with high school algebra and geometry, at least indirectly,

(3) differ from high school courses in content,

(4) coördinate other courses and fill in gaps.

Professor N. A. Court of the University of Oklahoma has developed a course

called college geometry, synthetic in character with mostly euclidean, but some projective theorems. The mimeographed sheets of his course were used as a basis for part of a teachers' course at Northwestern University last year. The latter part of this paper dealt with the lack of training in synthetic geometry in many schools and outlined the course developed by Professor Court.

2. Professor Curtiss' paper suggested definitions and proofs for the integration of functions of a complex variable which, if not new, differ in some respects from the conventional ones. Some demonstrations were simplified by the use of the indefinite integral. Cauchy's integral formula was especially considered. For polynomials this formula is a direct corollary of the divisibility of $f(t) - f(z)$ by $(t - z)$. This proof was extended to the general case without the use of infinite series, and examples were given tending to throw light on the meaning and applicability of the formula. Cauchy's inequality and Liouville's theorem were discussed and the latter was deduced from the former.

3. The National Research Council has authorized the preparation and publication of certain reports on mathematics. Professor Coble's paper was devoted to a discussion of some of the problems considered by a committee in connection with such a report in the field of algebraic geometry.

4. Miss Beaty read a paper which served as an introduction for further discussion on the advantages and disadvantages of unified courses in freshman work. Unified courses seem to present the field of mathematics in a way which develops a thorough understanding, a broader vision and a more comprehensive knowledge than do the separate systems. In her paper she cited as authorities, various statements made by professors throughout the country and all seem to favor the idea of unification especially for students who take only the required amount of mathematics in their college work.

G. H. SCOTT, *Secretary-Treasurer.*

EIGHTH ANNUAL MEETING OF THE KENTUCKY SECTION.

The eighth annual meeting of the Kentucky Section was held at Centre College, Danville, Ky., on Saturday, May 17, 1924. The chairman, Professor A. R. Fehn, of Centre College, presided. Between the morning and afternoon sessions Centre College entertained at a luncheon those in attendance. At the close of the luncheon Dr. Ames Montgomery, president of Centre College, made a few brief remarks welcoming the Section.

Among the twenty-five in attendance were the following six members of the Association: P. P. Boyd, J. M. Davis, H. H. Downing, A. R. Fehn, F. Elizabeth Le Stourgeon, E. L. Rees.

Professor J. M. DAVIS, University of Kentucky, and Professor A. R. FEHN, Centre College, were elected chairman and secretary-treasurer, respectively, for 1924–25. The meeting in 1925 will be held at the University of Kentucky.

The following papers were read:

(1) "Archimedes' solution derived from mechanics" by Dean P. P. BOYD, University of Kentucky.

(2) "Distribution of the zeros of Bessel's functions" by Professor GUY STEVENSON, Georgetown College (by invitation).

(3) "A vector proof of the theorem of Coriolis" by Professor E. L. REES, University of Kentucky.

(4) "An investigation concerning mathematics" by Professor HENRY LLOYD, Transylvania College (by invitation).

(5) "Manipulation versus understanding in mathematics" by Professor W. B. HUGHES, Kentucky Wesleyan College (by invitation).

(6) "Gauss' rule for Easter" by Professor H. H. DOWNING, University of Kentucky.

Abstracts of the papers follow:

1. Professor Boyd's talk was based on the article by Archimedes discovered by Dr. J. L. Heiberg in Constantinople in 1906, and which was published in the *Monist* of April, 1909. This article, with introduction by Professor D. E. Smith, was reprinted by the Open Court Company. An account of Archimedes' mathematical work was given by the speaker and examples of his solutions by means of mechanics were presented. A characteristic demonstration of Cavalieri's was added to illustrate the early origin of those concepts that immediately preceded the introduction of the calculus.

2. Professor Stevenson gave a brief history of Bessel's differential equation and of the Bessel functions. His paper included a short digest of the theorems of Bôcher on the relative positions of the zeros of J_n and J_{n+k}, for $k = 1$ and $k = 2$ and for $2p < k \leqq 2p + 2$. A few graphs of J_n were shown for some special values of the parameter n.

3. Professor Rees gave a new vector proof of the theorem of Coriolis, using the Gibbs' notation. This proof affords a good example of the economy of the vector notation and of the simplicity and brevity of vector methods in the treatment of kinematics.

4. Professor Lloyd gave some results of a questionnaire that he had sent to numerous colleges and universities. The questions related to the student's preparation upon entering college, his interest in his work while in college, the teacher's attitude, and the various features involved. Professor Lloyd has received many answers and in his review of these many interesting things were brought out.

5. Professor Hughes emphasized the importance of mathematics for the training of the faculty of reason. The formulas, rules and tables are the mere tools of the subject and the rapid use of these in manipulation should not be mistaken for an understanding of the underlying principles. If more time were spent in the grades and high school in getting a clear insight into the fundamentals of the subject, students would enter college with a very different attitude. The college teacher can assist by going back and helping the student to understand fundamentals passed over in the lower grades and he can point out to

prospective teachers the importance of securing a correct understanding of these on the part of the pupils.

6. Professor Downing employed Gauss' rule in the determination of the dates of Easter for certain years. The reasons for using the remainders obtained by dividing the year number, the century number, and certain other quantities by various divisors, were given. The two exceptional cases due to the interval between two successive full moons were mentioned and the modifications to the usual form of the rule given. H. H. DOWNING, *Acting Secretary.*

THE APRIL MEETING OF THE MICHIGAN SECTION.

The first meeting of the Michigan Section of the Mathematical Association of America was held at the University of Michigan, Ann Arbor, Michigan, on April 3, 1924, in conjunction with the Michigan Academy of Science, with Chairman T. H. Hildebrandt presiding.

The attendance was fifty-seven, including the following thirty-six members of the Association:

N. H. Anning, J. F. Barnhill, H. Blair, P. N. Blessing, C. C. Craig, A. Darnell, B. F. Dostal, W. W. Denton, L. C. Emmons, J. P. Everett, Florence E. Field, P. Field, W. B. Ford, E. F. Gee, J. W. Glover, V. G. Grove, L. A. Hopkins, T. H. Hildebrandt, L. C. Karpinski, D. Kazarinoff, T. Lindquist, C. E. Love, S. W. Mullen, A. N. Nelson, H. L. Olson, W. H. Pearce, O. J. Peterson, L. C. Plant, V. C. Poor, C. Reid, C. H. Richardson, L. J. Rouse, T. R. Running, R. C. Shellenbarger, E. R. Sleight, W. J. Thome.

After three papers of the program had been presented a business session resulted in the following transactions:

The minutes of the organization meeting of the Section were read and approved. The secretary reported a section membership of fifty out of a possible seventy-four members of the Mathematical Association from Michigan.

Chairman Hildebrandt presented an invitation from the executive officers of the Michigan Academy of Science for this section to hold its meetings regularly under the auspices of the Academy. The following motion was unanimously adopted:

Resolved that the Michigan Section of the Mathematical Association of America approve of the action of the Executive Committee in accepting the invitation of the Michigan Academy of Sciences to meet with it at this time; that this Section expresses itself as favoring the organization of a Mathematics Section of the Academy, the understanding being that the chairman of the section of the Association be automatically the chairman of the section of the Academy, and that the two bodies meet as one, whenever the meetings of the Association coincide in time and place with those of the Academy.

To comply with the constitutional provisions of the Academy of Science in recognizing new sections, a petition asking for such recognition was drawn up,

signed by about twenty members of the Section, and left in the hands of the chairman to be transmitted to the Academy.

A motion was carried to the effect that the chairman appoint a nominating committee to present names of officers for the ensuing year. The chairman appointed as this committee, W. B. FORD, H. BLAIR, and A. DARNELL. A little later the committee reported, recommending as officers the following persons: Chairman, E. R. SLEIGHT, Albion College, Albion; Secretary-Treasurer, J. P. EVERETT, Western Normal College, Kalamazoo; Member of the Executive Committee, N. H. ANNING, University of Michigan, Ann Arbor. Upon motion, which carried, the secretary was instructed to cast the unanimous ballot of the Section in accordance with the report of the nominating committee, and the persons named therein were declared elected.

The Section adjourned soon after noon and enjoyed a joint luncheon with the Mathematics Section of the Michigan Schoolmasters' Club.

The following papers were presented:

(1) "The theory of sampling" by Professor H. C. CARVER, University of Michigan (by invitation).

(2) "Some configurations associated with a net on a surface" by Professor V. G. GROVE, Michigan Agricultural College.

(3) "On the hatchet planimeter" by Mr. D. KAZARINOFF, University of Michigan.

(4) "Early American arithmetics" by Professor L. C. KARPINSKI, University of Michigan.

(5) "Some problems in yield in land contracts" by Professor L. C. EMMONS, Michigan Agricultural College.

(6) "A theorem on means" by Professor N. H. ANNING, University of Michigan.

Abstracts of papers, numbered as in the above list, follow:

(1) Professor Carver outlined the theory of sampling, of which the theory of least squares is a special case. The development showed that the formula for the probable error of the mean should have the factor $n - 1$ in the denominator instead of n, and that the choice of $n - 1$ was not arbitrary. Moreover, the probable error of the mean of a series of observations that are not symmetrically arranged about their mean is quite as significant as though they lacked skewness. He mentioned the fact that the correct formula for the probable error of the mean is necessarily an infinite series, and has as yet never been developed. This arises from the fact that the numerator of the formula generally employed contains as a factor the so-called dispersion or measure of precision, and that this dispersion is not the dispersion of the sample but rather that of the entire population from which the sample is drawn. In other words, the probable error of the mean is a function of a "second moment," which is itself subject to a probable error. This probable error is a function of higher "moments," and so on ad infinitum. The resulting infinite series should be investigated. Until it can be shown that the series is convergent, we have no exact idea of the nature of

probable error. In fact, if the population from which the sample be drawn is strictly "normal," the distribution of higher moments must be significantly skew, and the problem is almost as difficult as though no restrictions as to skewness were made.

(2) Professor Grove defined the associate conjugate and the reflected associate nets associated with a general net. The usual definition of the ray (axis) of a point with respect to a net was given, thus giving rise to three rays (axes), one for each of the three nets. If any two of these rays (axes) coincide, all three do. There exists one and only one pair of Green reciprocal lines in Green's relation R with respect to the general net. If two of the rays (axes) coincide with the plane (space) component of this pair, then the net is self-dual.

(3) Mr. Kazarinoff explained the mechanical construction and general theory of the hatchet planimeter and showed that it might be employed with a relative error as small as one per cent.

(4) Professor Karpinski illustrated his paper with projections of photographic reproductions taken from early American arithmetics. The first arithmetical work published in the new world was written in Spanish and published at Mexico City by Juan Diez Freyle in 1556. The first Colonial arithmetic appeared in 1705; it was entitled, "*The Secretary's Guide & Young Man's Companion*," published in New York by William Bradford. About one hundred twenty-five works on arithmetic were published in America before 1800, of which the most popular was written by Thomas Dilworth and reprinted many times from 1770 to 1825. The next in popularity was another English work, "*The Young Man's Companion*," written by George Fisher, of which probably twenty editions appeared before 1800. Professor Karpinski also showed a number of slides portraying Maya and Aztec methods of representing numbers.

(5) Professor Emmons discussed the construction of tables for determining maturity dates of land contracts upon which fixed monthly payments are made. He showed how these tables might be used to solve a variety of problems, including the following: to determine the payment necessary to pay off a contract in any given time; to find the interest yield on a contract that is sold at a discount; to find the yield on a similar contract where a straight mortgage is involved; and to find the yield when a mortgage requiring fixed payments at equal intervals is involved.

(6) A comprehensive definition of a mean was set up by Professor Anning. With its aid the following theorem was proved:

THEOREM: *The mean of n quantities is the same as the mean of the ${}_nC_k$ means formed by taking $(n-k)$ of the quantities in all possible ways*, $(1 \leq k \leq n-1)$.

Some elementary applications were mentioned.

J. P. EVERETT, *Secretary-Treasurer.*

THE MAY MEETING OF THE MINNESOTA SECTION.

The regular spring meeting of the Minnesota Section was held at Hamline University, St. Paul, on Saturday, May 24, 1924. There was an afternoon session followed by dinner in the Hamline University Refectory. Professor H. H. Dalaker, chairman of the Section, presided.

The attendance was thirty-five including the following thirteen members of the Association: W. O. Beal, R. W. Brink, W. H. Bussey, H. H. Dalaker, O. C. Edwards, Clara L. Hancock, C. A. Herrick, D. Jackson, R. A. Johnson, E. L. Mickelson, A. L. Underhill, W. R. Warne, H. B. Wilcox.

The following officers were elected for the coming year: Chairman, W. H. KIRCHNER, College of Engineering, University of Minnesota; Secretary, A. L. UNDERHILL, College of Science, Literature and the Arts, University of Minnesota; Executive Committee, R. A. JOHNSON, Hamline University, CLARA L. HANCOCK, Virginia Junior College, G. C. PRIESTER, University of Minnesota. A motion was passed, expressing the appreciation of the section for the hospitality of Hamline University in entertaining the Section.

The following six papers were read:

(1) "The numbers of Fibonacci and a problem in algebraic equations" by Professor R. A. JOHNSON, Hamline University.

(2) "On the pressure due to wave action" by Professor G. C. PRIESTER, University of Minnesota.

(3) "An explosion transformation" by Mr. C. A. RUPP, Hamline University.

(4) "The trigonometry of correlation" by Professor DUNHAM JACKSON, University of Minnesota.

(5) "An example of numerical differentiation" by Professor W. O. BEAL, University of Minnesota.

(6) "Parametric representations of algebraic curves" by Professor H. H. DALAKER, University of Minnesota.

Abstracts of the papers follow:

(1) The sequence of numbers 1, 1, 2, 3, 5, 8, $\cdots$ defined by the relation $u_n = u_{n-1} + u_{n-2}$, $u_0 = 0$, $u_1 = 1$, has a number of interesting and amusing properties. Mr. Johnson discussed some of the more obvious of these, and showed their connection with the solution of the simultaneous equations $x_1x_2 = x_3$, $x_2x_3 = x_4$, $\cdots$ $x_{n-2}x_{n-1} = x_n$, $x_{n-1}x_n = x_1$, $x_nx_1 = x_2$.

(2) Professor Priester's paper gives a method of determining the hydrostatic head at the crest and trough of a trochoidal wave. The fundamental theory is based on Gerstner's equations for trochoidal waves and by means of the usual dynamical equations of motion the equation of pressure is developed and expressed in logarithmic form rather than the common exponential form. By plotting certain functions of this equation it is possible to determine the hydrostatic head for any length and height of wave and at any depth below the free surface directly instead of by the usual method of trial. The necessary graphs and the solution of a specific problem are incorporated in the paper.

(3) The explosion transformation discussed by Mr. Rupp consists in exploding the points of an arbitrary plane curve into circles in that plane having the exploded point as center, and an arbitrary point function as radius. Several cases were discussed. If the radius is constant, we have a dilation. If the radius is proportional to the distance of the point from a fixed pole, the transform of a straight line is a conic whose eccentricity is the constant of proportionality. If the radius is equal to the distance of the exploded point from the pole, so that all the circles pass through a common point, the transform is the podoid of the original curve. Several applications were given.

(4) Professor Jackson's paper appeared in full in the June number of the MONTHLY.

(5) In 1888, G. W. Hill obtained by mechanical quadrature a numerical tabular solution of the differential equations of motion of Hyperion considered as attracted by both Saturn and its satellite Titan, the latter being assumed to move in a circular orbit. Mr. Beal has computed the derivatives of the coördinates of Hyperion for each of the arguments in the table, and exhibited how well the differential equations of motion and their Jacobian integral are satisfied. The first derivatives of the coördinates were also shown to satisfy the differential "equations of variation" arising when Titan is assumed to move in an ellipse.

(6) In his paper Professor Dalaker gave a brief discussion of the parametric representations of algebraic curves, together with examples of the different classes of functions necessary for the uniformization of curves of genus *zero*, *one* and *two*.

R. W. BRINK, *Secretary-Treasurer.*

ORGANIZATION MEETING OF THE NEBRASKA SECTION.

A meeting for the purpose of organizing a Nebraska Section of the Association was held at Lincoln March 14, 1924. Miss Emma Hanthorn of the State Teachers College at Kearney presided. After a brief discussion by Dr. W. C. Brenke of the State University outlining the history and purposes of the Association, and the policy of the MONTHLY, an executive committee was chosen to draw up the by-laws.

The newly formed section was called to meet jointly with the Mathematics-Physics Section of the Nebraska Academy of Sciences which met at Creighton College, Omaha, May 2, 1924. The Executive Committee presented the by-laws which were adopted and signed by the following charter members:

Harriet Anderson, Grand Island College; W. C. Brenke, University of Nebraska; A. L. Candy, University of Nebraska; A. R. Congdon, University of Nebraska; M. G. Gaba, University of Nebraska; Emma E. Hanthorn, Kearney Teachers College; Mary F. Jackson, Lincoln High School; Nellie M. Johnston, Gibbon High School; Stella B. Kirker, Lincoln High School; E. P.

Martinson, University of Nebraska; R. M. McDill, Hastings College; J. E. Opp, University of Nebraska; T. A. Pierce, University of Nebraska; T. I. Porter, University of Omaha; W. T. Rigge, Creighton University; C. R. Sherer, University of Nebraska.

The following officers were elected for the ensuing year: Chairman—W. C. Brenke; Secretary-Treasurer—Emma E. Hanthorn; Member Executive Committee—R. M. McDill.

Papers read at the first annual meeting of the Nebraska Section of the Mathematical Association of America, meeting jointly with the Mathematics-Physics Section of the Academy of Sciences, with W. C. Brenke presiding, were as follows:

(1) "A function approximating the least root of an equation," Prof. T. A. Pierce, University of Nebraska. (By title.)

(2) "The expansion of the cube root of an integer into a continued fraction," Mr. A. E. Anderson, University of Nebraska.

(3) "Some problems in geometry," Prof. M. G. Gaba, University of Nebraska.

(4) "Two problems," Prof. R. M. McDill, Hastings College.

(5) "A compound harmonic motion machine" (illustrated), Prof. W. H. Rigge, Creighton University.

(6) "An angle connected with the mean anomaly in elliptic orbits," Prof. W. C. Brenke, University of Nebraska.

(7) "An explicit solution for a general congruence with prime modulus," Prof. Pierce. (By title.)

(8) "A space transformation," Mr. E. P. Martinson, University of Nebraska.

(9) "An example of the value of diophantine analysis to the teacher in the secondary school," Mr. A. E. Campbell, Omaha Technical High School.

Emma E. Hanthorn, *Secretary-Treasurer.*

A THEOREM ON ISOGONAL TETRAHEDRA.

By B. H. BROWN, Dartmouth College.

1. Introduction. It has often been remarked how greatly our present knowledge of the geometry of the triangle exceeds that of the tetrahedron, and various excellent reasons may be cited therefor. I think this superiority likely to continue, since new methods for the study of the tetrahedron may well make even more important contributions to the geometry of the triangle. Such was the case with the space locus for the fourth vertex of an involutory tetrahedron of fixed base, discussed by Neuberg in his remarkable *Mémoire sur le tétraèdre.*[1] From this locus he deduces the theorem *in plano,* that for the general triangle the following 21 notable points lie on a circular cubic:

[1] Published as *Supplément V* to *Mathesis,* 1885, also *Mémoires couronnés,* etc., Belgium, 1886, pp. 1–72.

(*a*) the 3 vertices A_1, A_2, A_3,
(*b*) the 3 points symmetric to the vertices in the opposite sides a_1, a_2, a_3,
(*c*) the 6 vertices of the equilateral triangles constructed on a_1, a_2, a_3,
(*d*) the 3 points symmetric to A_1, A_2, A_3 in the lines $q_{23}q_{32}$, $q_{31}q_{13}$, $q_{12}q_{21}$ respectively, where q_{ns} is the point where the side a_n meets the perpendicular bisector of a_s,
(*e*) the circumcenter,
(*f*) the orthocenter,
(*g*) the 2 isodynamic centers,[1]
(*h*) the 2 isogonal centers.[2]

The development of this theorem is long but not difficult. Still, the chain of theorems with which we are concerned, although entirely elementary, cannot be casually referred to as "well-known." It seems desirable to summarize the pertinent portion of the memoir.

Definition. If the six planes are constructed perpendicular to the mid-points of the six edges of a tetrahedron, and if the six points in which these planes meet the opposite edges are coplanar, the tetrahedron is said to be involutory.

The necessary and sufficient (N. S.) condition for an involutory tetrahedron is

$$(a_1^2 - a_6^2)(a_2^2 - a_4^2)(a_3^2 - a_5^2) = (a_1^2 - a_5^2)(a_2^2 - a_6^2)(a_3^2 - a_4^2), \qquad (1)$$

or

$$\begin{vmatrix} 1 & a_1^2 + a_4^2 & a_1^2 a_4^2 \\ 1 & a_2^2 + a_5^2 & a_2^2 a_5^2 \\ 1 & a_3^2 + a_6^2 & a_3^2 a_6^2 \end{vmatrix} = 0; \qquad (2)$$

where a_1, a_2, a_3 are the lengths of the three edges of a base, and a_4, a_5, a_6 the lengths of the edges opposite a_1, a_2, a_3 respectively.

If the base of a tetrahedron be fixed, and if the fourth vertex vary so that the tetrahedron remain involutory, this fourth vertex has in general two degrees of freedom; its locus is consequently a surface, which Neuberg has shown to be a circular cubic surface. The circular cubic curve in which the plane of the base triangle is met by this surface is the cubic previously referred to. Neuberg finds various notable one-parameter families of involutory tetrahedra with this base, finds the space locus (a curve) for the fourth vertex; the intersection of this curve with the plane of the base gives a point or points on the cubic. We shall limit ourselves to the most interesting cases (*f*), (*g*), and (*h*), and approach these from the following standpoint.

In general, in a tetrahedron each of the following quadruples of lines belongs to one set of rulings of a hyperboloid: (*f*) the four altitudes, (*g*) the four lines

[1] The two points W such that

$$A_1A_2 \cdot A_3W = A_1A_3 \cdot A_2W = A_1W \cdot A_2A_3.$$

Each of A_1, A_2, A_3, W is an isodynamic center of the other three.

[2] The two points Z from which the sides a_1, a_2, a_3 subtend angles of 60° or of 120°. *Cf.* Lufkin, MONTHLY (*1923*, 127–131).

from the vertices to the centers of the circles inscribed in the opposite faces, (*h*) the four lines from the vertices to the points of contact of the opposite faces with the inscribed sphere. A tetrahedron in which any such quadruple are concurrent is called: (*f*) orthogonal, (*g*) isodynamic, (*h*) isogonal.

(*f*) The N.S. condition for an orthogonal tetrahedron is

$$a_1^2 + a_4^2 = a_2^2 + a_5^2 = a_3^2 + a_6^2, \tag{3}$$

hence from (2) Neuberg observes that every orthogonal tetrahedron is involutory. The space locus here is a line perpendicular to the plane of the base of the tetrahedron at the orthocenter of the base triangle, which is therefore a notable point on the cubic.

(*g*) The N.S. condition for an isodynamic tetrahedron is

$$a_1a_4 = a_2a_5 = a_3a_6, \tag{4}$$

hence from (2) Neuberg observes that every isodynamic tetrahedron is involutory. The space locus here is a circle perpendicular to the plane of the base triangle with the isodynamic centers as diametral points, consequently the isodynamic centers are notable points on the cubic.

(*h*) Here we find a striking lacuna in Neuberg's treatment. He does indeed show that the space locus for the fourth vertex of isogonal tetrahedra with fixed base is a pair of hyperbolas each with vertices at an isodynamic and at an isogonal center of the triangle. But the evidence is conclusive that he never even suspected that isogonal tetrahedra were involutory. Yet from other considerations he evidently thought it conceivable that the isogonal centers are notable points on the cubic, and he proves they are, the proof being long and circuitous.

2. The Missing Theorem. After this tedious but necessary introduction we may state the main purpose of this paper which is simply to prove the

THEOREM: *Every isogonal tetrahedron is involutory.*

Proof. It is a characteristic property of an isogonal tetrahedron that lines from any point of tangency of the inscribed sphere to the vertices in that face make angles of 120° with each other.[1] Hence by the cosine law

$$\begin{aligned}
a_1^2 &= \rho_2^2 + \rho_3^2 + \rho_2\rho_3 \\
a_2^2 &= \rho_3^2 + \rho_1^2 + \rho_3\rho_1 \\
a_3^2 &= \rho_1^2 + \rho_2^2 + \rho_1\rho_2 \\
a_4^2 &= \rho_1^2 + \rho_4^2 + \rho_1\rho_4 \\
a_5^2 &= \rho_2^2 + \rho_4^2 + \rho_2\rho_4 \\
a_6^2 &= \rho_3^2 + \rho_4^2 + \rho_3\rho_4,
\end{aligned} \tag{5}$$

[1] In every tetrahedron the three angles at the points of tangency are the same in each face, a theorem apparently due to Bang, *Tidsskrift for Math.*, 1897, p. 48; solved by Gehrke, *ibid.*, p. 84; extended by Meyer, *Jahresbericht*, 1903, vol. 12, p. 137. In the isogonal tetrahedron the Bang angles are equal.

where ρ_i is the tangential distance from A_i. Neuberg was "content with proposing"[1] the elimination of the ρ's. Now

$$\begin{cases} a_1^2 - a_6^2 = (\rho_2 - \rho_4)(\rho_2 + \rho_3 + \rho_4) \\ a_2^2 - a_4^2 = (\rho_3 - \rho_4)(\rho_3 + \rho_4 + \rho_1) \\ a_3^2 - a_5^2 = (\rho_1 - \rho_4)(\rho_4 + \rho_1 + \rho_2) \end{cases}$$

$$\begin{cases} a_1^2 - a_5^2 = (\rho_3 - \rho_4)(\rho_2 + \rho_3 + \rho_4) \\ a_2^2 - a_6^2 = (\rho_1 - \rho_4)(\rho_3 + \rho_4 + \rho_1) \\ a_3^2 - a_4^2 = (\rho_2 - \rho_4)(\rho_4 + \rho_1 + \rho_2), \end{cases}$$

and we see immediately that (1) is a first eliminant of (5).[2]

From Neuberg's results we may therefore conclude immediately:

THEOREM: *The isogonal centers are notable points on the cubic.*

3. Degenerate Case. It is of interest to note the degeneration when the base triangle is equilateral. Those of the 21 points which are determinate reduce to 7, to which we may adjoin the circular points at infinity. However, there is no unique cubic herewith associated, for this turns out to be one of those groups of 9 points through which pass a pencil of cubics. This could have been foreseen in two different ways. First, as Lufkin (*1923*, 128) pointed out, in an equilateral triangle one isogonal point may be anywhere on the circumcircle of the triangle. Second, every tetrahedron with an equilateral base is involutory. Consequently the space locus of the fourth vertex is all finite space except for this plane, and hence every finite point of the plane is continuously accessible.

4. A Second Eliminant. From a suggestion of my colleague, Professor Wilder, I have succeeded in finding a second eliminant of the six equations (5). The following equations are easily verified:

$$\begin{bmatrix} - a_4^4a_3^2 - a_5^4a_3^2 + a_4^4a_2^2 + a_5^4a_1^2 - a_4^2a_2^4 \\ - a_5^2a_2^4 - a_3^2a_2^4 - a_4^2a_1^4 - a_5^2a_1^4 - a_1^4a_3^2 \\ - a_4^2a_5^2a_2^2 - a_4^2a_5^2a_1^2 - 2a_5^2a_2^2a_3^2 + 2a_5^2a_1^2a_3^2 \\ + 2a_4^2a_5^2a_3^2 + 2a_4^2a_2^2a_3^2 - 2a_4^2a_1^2a_3^2 \\ + 2a_4^2a_2^2a_1^2 + 2a_5^2a_1^2a_2^2 + 2a_1^2a_2^2a_3^2 \end{bmatrix} \tag{6}$$

$$= (\rho_4 - \rho_3)(\rho_1 - \rho_2)^2(\rho_4^2 + \rho_1^2 + \rho_2^2 - \rho_4\rho_1 - \rho_4\rho_2 - \rho_1\rho_2)(\rho_1 + \rho_2 + \rho_3),$$

$$a_4^2 - a_5^2 = (\rho_1 - \rho_2)(\rho_1 + \rho_2 + \rho_4), \tag{7}$$

$$\begin{pmatrix} a_2^2a_5^2 - a_1^2a_4^2 - a_2^2a_3^2 \\ + a_1^2a_3^2 + a_4^2a_3^2 - a_5^2a_3^2 \end{pmatrix} \tag{8}$$

$$= (\rho_1 + \rho_2 + \rho_3)(\rho_1 - \rho_2)(\rho_4 - \rho_3)(\rho_4 + \rho_1 + \rho_2),$$

[1] *Mémoire sur le tétraèdre*, p. 52 in the Belgian memoirs; see also *Jahresbericht*, 1907, vol. 16, p. 358. The reader will also find another unsolved algebraic problem relating to isogonal tetrahedra in these articles.

[2] If a tetrahedron is isogonal to the inscribed sphere it is isogonal to the seven escribed spheres; that is, the corresponding quadruples of lines are concurrent. The proofs that (1) is an eliminant for escribed isogonality are unnecessary, but the reader will find them interesting variants on the proof given.

$$(a_4^4 - a_4^2a_5^2 + a_5^4 - a_3^2a_5^2 - a_4^2a_5^2 + a_3^4) = (\rho_4 + \rho_1 + \rho_2)^2(\rho_4^2 + \rho_1^2 + \rho_2^2 - \rho_4\rho_1 - \rho_4\rho_2 - \rho_1\rho_2), \tag{9}$$

$$(a_2^2 - a_1^2)^2 = (\rho_1 - \rho_2)^2(\rho_1 + \rho_2 + \rho_3)^2, \tag{10}$$

$$(a_1^2 + a_4^2 - a_2^2 - a_5^2)^2 = (\rho_1 - \rho_2)^2(\rho_4 - \rho_3)^2. \tag{11}$$

Now the product of the right-hand members of (6), (7), and (8) is equal to the product of the right-hand members of (9), (10), and (11). Hence equating the product of the left-hand members of the first group to that of the second group, we can easily verify that we actually have an eliminant, and that it is independent of (1) since it does not contain a_6.

5. Associated Unsolved Problems. The other problem proposed by Neuberg, reference to which was made in a foot-note, is this. If f_i denote the area of the face of an isogonal tetrahedron opposite A_i, and if t_{ij} denote the area of either of the two equal small triangles bounded by an edge a_{ij} and by lines from vertices A_i and A_j to the points of tangency of the inscribed sphere with the faces opposite A_k or A_l, then

$$\begin{cases} t_{12} + t_{23} + t_{31} = f_4 \\ t_{23} + t_{34} + t_{42} = f_1 \\ t_{34} + t_{41} + t_{13} = f_2 \\ t_{41} + t_{12} + t_{24} = f_3 \\ t_{23}t_{41} = t_{31}t_{42} = t_{12}t_{43}, \end{cases}$$

the last series of equalities characterizing isogonal tetrahedra. Solve for the t's in terms of the f's. The reader will note that an equivalent problem is to find the edges of an isodynamic tetrahedron in terms of the perimeters of the faces.

Finally, what geometric significance attaches to this second eliminant? Can the eliminants be simplified or symmetrized? What is the geometric significance of the eccentricities of the hyperbolas mentioned above? What is the significance of the real infinite point on the cubic with respect to the base triangle? Are there not other notable points to be found from other types of involutory tetrahedra?

UNIQUENESS OF THE LORENTZ TRANSFORMATION.

By ALONZO CHURCH, Princeton University.

1. The object of this inquiry is to obtain a set of logically independent postulates which uniquely determine the Lorentz transformation for one dimension. For this purpose we propose the following set:

1. The required transformation expresses $\bar{x}$ and $\bar{t}$ as functions of x, t, and v, which have continuous partial derivatives with respect to x and t, where $\bar{x}$, $\bar{t}$, x, and t are real variables, and v is a parameter which may have any value less than 1 and greater than -1.

2. For no value of v is the partial derivative of $\bar{t}$ with respect to t negative for every value of x and t.

3. $dx/dt = 1$ implies that $d\bar{x}/d\bar{t} = 1$.

4. $dx/dt = v$ implies that $d\bar{x}/d\bar{t} = 0$.

5. The inverse transformation, which expresses x and t in terms of $\bar{x}$ and $\bar{t}$, is obtained from the direct transformation by replacing $\bar{x}$ by x, and $\bar{t}$ by t, and x by $\bar{x}$, and t by $\bar{t}$, and v by $-v$.

6. The transformation is unchanged if $\bar{x}$ be replaced by $-\bar{x}$, and x by $-x$, and v by $-v$.

If we suppose our units of measurement so chosen that the velocity of light is 1, the physical meaning of the last five of these postulates is as follows:

2. The transformed time, $\bar{t}$, does not flow backwards with respect to t.

3. The velocity of light is invariant.

4. The origin of the transformed coördinate system is moving along the x-axis with velocity v in the original coördinate system.

5. The resultant of two velocities with the same numerical value and with opposite directions is zero.

6. The form of the transformation is independent of our choice of a positive direction on the x-axis.

2. Let the required transformation have the form:

$$\bar{x} = \varphi(x, t, v), \qquad \bar{t} = \psi(x, t, v).$$

Then

$$\left(\psi_x \frac{dx}{dt} + \psi_t\right)\frac{d\bar{x}}{d\bar{t}} = \varphi_x \frac{dx}{dt} + \varphi_t. \tag{1}$$

Setting $dx/dt = v$ in (1), it follows from postulate 4 that

$$v\varphi_x + \varphi_t = 0. \tag{2}$$

It is a consequence of postulates 4 and 5 that $dx/dt = 0$ implies that $d\bar{x}/d\bar{t} = -v$. Therefore, setting $dx/dt = 0$ in (1), it follows that

$$\varphi_t + v\psi_t = 0. \tag{3}$$

Integrating this with respect to t, we obtain

$$\varphi + v\psi = X, \tag{4}$$

where X is independent of t.

Setting $dx/dt = 1$ in (1), it follows from postulate 3 that

$$\psi_x + \psi_t = \varphi_x + \varphi_t. \tag{5}$$

From (2) and (3) it follows that

$$\varphi_x = \psi_t \tag{6}$$

and therefore by (5)

$$\psi_x = \varphi_t \tag{7}$$

Substituting in (2) this gives

$$v\varphi_x + \psi_x = 0$$

and therefore integrating with respect to x,

$$v\varphi + \psi = T, \tag{8}$$

where T is independent of x.

For values of v between $+1$ and -1 equations (4) and (8) can be solved simultaneously. In this way we obtain:

$$\varphi = X_1 + T_1, \qquad \psi = X_2 + T_2,$$

where X_1 and X_2 are independent of t, and T_1 and T_2 are independent of x.

From equations (6) and (7) it now follows that

$$\frac{dX_1}{dx} = \frac{dT_2}{dt}, \qquad \frac{dX_2}{dx} = \frac{dT_1}{dt}.$$

These conditions, however, can be satisfied only if dX_1/dx, dX_2/dx, dT_1/dt, and dT_2/dt are all independent of both x and t. Therefore X_1, X_2, T_1, and T_2 are of the first degree in x and t. Therefore φ and ψ are of the first degree in x and t. Let us accordingly write the transformation in the form,

$$\bar{x} = p_1x + q_1t + r_1, \qquad \bar{t} = p_2x + q_2t + r_2,$$

where p_1, p_2, q_1, q_2, r_1, and r_2 are functions of v.

Then, by (6)

$$p_1 = q_2,$$

by (7)

$$p_2 = q_1,$$

and by (3)

$$q_1 = -vq_2.$$

Therefore

$$p_2 = -vq_2.$$

The required transformation can therefore be written in the form

$$\bar{x} = \beta(x - vt) + r_1, \qquad \bar{t} = \beta(t - vx) + r_2, \tag{9}$$

where β, r_1, and r_2 are functions of v. In order that postulate 6 be satisfied it is necessary that β and r_2 be even functions of v, and that r_1 be an odd function.

By solving (9) for x and t we obtain

$$x = \frac{\bar{x} + v\bar{t}}{(1 - v^2)\beta} - \frac{r_1 + vr_2}{(1 - v^2)\beta}, \qquad t = \frac{\bar{t} + v\bar{x}}{(1 - v^2)\beta} - \frac{r_2 + vr_1}{(1 - v^2)\beta}.$$

Consequently postulate 5 requires that

$$r_1 = \frac{r_1 + vr_2}{(1 - v^2)\beta}, \qquad r_2 = -\frac{r_2 + vr_1}{(1 - v^2)\beta}, \qquad \beta = \frac{1}{(1 - v^2)\beta}.$$

Solving these equations simultaneously for β, r_1, and r_2, we find that

$$r_1 = 0, \qquad r_2 = 0, \qquad \beta = \pm \frac{1}{\sqrt{1 - v^2}}.$$

The expression obtained for β represents, not two but indefinitely many determinations of β as a function of v, since it is possible to take the upper sign for some values of v and the lower sign for the remaining values, in any arbitrary manner which makes β an even function of v. It is for this reason that postulate 2 is required.

In accordance with postulate 2, there is only one possible determination of β, namely,

$$\beta = \frac{1}{\sqrt{1 - v^2}}.$$

The required transformation, therefore, has the form:

$$\bar{x} = \frac{x - vt}{\sqrt{1 - v^2}}, \qquad \bar{t} = \frac{t - vx}{\sqrt{1 - v^2}},$$

which is the Lorentz transformation for one dimension.

3. We shall give independence proofs for the last five of our postulates. There is no immediately evident independence proof for the first postulate. It is quite possible that it is not necessary to assume the existence of the partial derivatives, φ_x, ψ_x, φ_t, ψ_t.

As an independence proof for postulate 2 we may cite the transformation:

$$\bar{x} = f(v)\frac{x - vt}{\sqrt{1 - v^2}}, \qquad \bar{t} = f(v)\frac{t - vx}{\sqrt{1 - v^2}},$$

where $f(v)$ is a function of v which is restricted to have always one of the two values $+1$ and -1, in such a way that $f(v) = f(-v)$ for all values of v, and $f(v)$ is not equal to $+1$ for every value of v.

An independence proof for postulate 3 is the transformation:

$$\bar{x} = x - vt, \qquad \bar{t} = t.$$

The simplest independence proof for postulate 4 is:

$$\bar{x} = x, \qquad \bar{t} = t,$$

but we may also take the transformation:

$$\bar{x} = \frac{x - kvt}{\sqrt{1 - k^2v^2}}, \qquad \bar{t} = \frac{t - kvx}{\sqrt{1 - k^2v^2}},$$

where k is any constant which is real and less than 1 in numerical value.

An independence proof for postulate 5 is:

$$\bar{x} = x - vt, \qquad \bar{t} = t - vx.$$

And, finally, for postulate 6 we have:

$$\bar{x} = \frac{(1 - v)^n}{(1 + v)^{n+1}}(x - vt), \qquad t = \frac{(1 - v)^n}{(1 + v)^{n+1}}(t - vx), \tag{10}$$

where n is any real number different from $-\frac{1}{2}$.

4. The transformations (10) have all the essential properties of the Lorentz transformation except the property of symmetry expressed by postulate 6.

The transformation corresponding to $n = -\frac{1}{2}$ is the Lorentz transformation. The transformations corresponding to other values of n are arranged symmetrically about the value $-\frac{1}{2}$ in the sense that the transformation corresponding to $n = a$ and that corresponding to $n = -(a - 1)$ apply to the same space with a different choice of the positive direction on the x-axis.

In a space in which one of these transformations is valid there is an effect on the apparent length of a moving body analogous to that given by the Lorentz transformation, but this effect may be either a lengthening or a shortening, and, in accordance with the asymmetric character of the transformations, it depends not only on the speed of the motion but also on the direction.

The invariant of the transformations is:

$$\frac{(t - x)^{n+1}}{(t + x)^n}.$$

The proper time is given by

$$d\tau = \frac{(dt - dx)^{n+1}}{(dt + dx)^n}.$$

The transformation obtained by setting $n = 0$ in (10) is:

$$\bar{x} = \frac{x - vt}{1 + v}, \qquad \bar{t} = \frac{t - vx}{1 + v}.$$

This transformation has the properties just enumerated. The proper time given by it is $d\tau = dt - dx$. Since $\int (dt - dx)$ is independent of the path of integration, the absolute character of time is preserved in a way in which it is not by the Lorentz transformation. This means that there are no geodesics in the two-dimensional space-time to which this transformation applies; and that a unique proper time can be attached to every event, independently of a moving particle to which the time is referred.

5. We shall prove that the transformations (10) are the only transformations which satisfy postulates 1 to 5 above and the following three in addition:

7. $x = 0$, $t = 0$ implies that $\bar{x} = 0$, $\bar{t} = 0$.

8. The transformations corresponding to the various possible values of v constitute a group.

9. $\bar{x}$ and $\bar{t}$ are continuous as functions of v.

A transformation which satisfies these postulates must be of the form, (9), above. By postulate 7 we have $r_1 = r_2 = 0$. Therefore the transformation has the form:

$$\bar{x} = \beta(v)(x - vt), \qquad \bar{t} = \beta(v)(t - vx).$$

A second transformation of the set may have the form:

$$\bar{\bar{x}} = \beta(v')(\bar{x} - v'\bar{t}), \qquad \bar{\bar{t}} = \beta(v')(\bar{t} - v'\bar{x}).$$

Combining these two transformations, we obtain:

$$\bar{\bar{x}} = \beta(v)\beta(v')(1 + vv')\left(x - \frac{v + v'}{1 + vv'}t\right),$$

$$\bar{\bar{t}} = \beta(v)\beta(v')(1 + vv')\left(t - \frac{v + v'}{1 + vv'}x\right).$$

Therefore, in order that postulate 8 be satisfied, it is necessary that

$$\beta(v)\beta(v')(1 + vv') = \beta\left(\frac{v + v'}{1 + vv'}\right).$$

Let

$$\beta(v) = \frac{\varphi(v)}{1 + v}.$$

Then

$$\varphi(v)\varphi(v') = \varphi\left(\frac{v + v'}{1 + vv'}\right). \tag{11}$$

Now we know, by postulate 2, that $\varphi(v)$ is positive for some value of v between $+1$ and -1. Therefore, since it follows from postulate 9 that $\varphi(v)$ is continuous, there is some interval between $+1$ and -1 throughout which $\varphi(v)$ is positive. In order, therefore, to find the form of $\varphi(v)$ in this interval, we are justified in taking the logarithm of both sides of (11). It will turn out that the form of $\varphi(v)$ so obtained cannot vanish for any value of v between $+1$ and -1, from which it will follow that $\varphi(v)$ has this form throughout the interval from -1 to $+1$.

Taking the logarithm of each side of (11), we obtain:

$$\log \varphi(v) + \log \varphi(v') = \log \varphi\left(\frac{v + v'}{1 + vv'}\right).$$

Since v and v' lie between $+1$ and -1, we may make the following substitutions without going outside the real number system:

$$v = \tanh u$$
$$v' = \tanh w$$
$$\log \varphi(\tanh u) = F(u).$$

Making these substitutions, we obtain:

$$F(u) + F(w) = F(u + w). \tag{12}$$

Since u and w are independent variables, we may assign any relation between them that we please without affecting the validity of (12). Setting $u = w$, we have:

$$F(2w) = 2F(w).$$

Similarly, setting $u = 2w$,

$$F(3w) = 3F(w)$$

and so on. It is clear that we may prove by induction that

$$F(aw) = aF(w). \tag{13}$$

for every positive integer, a.

Setting $u = 0$ in (12), we obtain:

$$F(w) + F(0) = F(w).$$

Since $F(w)$ is not infinite for all values of w, it follows that

$$F(0) = 0.$$

Therefore, setting $u = -w$ in (12),

$$F(-w) = -F(w),$$

from which it follows that (13) is also true when a is a negative integer.

Setting $w = w'/a$ in (13), we obtain:

$$F\left(\frac{w'}{a}\right) = \frac{1}{a}F(w').$$

Therefore if p/q is any rational number,

$$F\left(\frac{p}{q}w\right) = \frac{p}{q}F(w),$$

and therefore, since $F(w)$ is continuous,

$$F(aw) = aF(w)$$

for every real number, a. And, dividing by aw,

$$\frac{F(w)}{w} = \frac{F(aw)}{aw}.$$

Therefore

$$\frac{F(w)}{w} = b,$$

or

$$F(w) = bw,$$

where b is an arbitrary constant. In order to make sure that b is actually arbitrary, we must, of course, substitute this expression for $F(w)$ in equation (12).

We now have

$$\varphi(v) = e^{b\tanh^{-1}v}$$

and therefore

$$\varphi(v) = \left(\frac{1-v}{1+v}\right)^n,$$

where n is an arbitrary constant. This gives

$$\beta(v) = \frac{(1-v)^n}{(1+v)^{n+1}},$$

from which it follows that the required transformation is in the form (10).

QUESTIONS AND DISCUSSIONS.

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

DISCUSSIONS.

I. INFINITE AND IMAGINARY ELEMENTS IN ALGEBRA AND GEOMETRY: REPLY TO CRITICISMS.

By R. M. WINGER, University of Washington.

Two criticisms of my article [1] have been printed in the MONTHLY, one by Professor Tomlinson Fort,[2] the other by Professor W. L. G. Williams.[3] Both these gentlemen express emphatic disagreement with my suggestions, partly on the grounds of pedagogy, but chiefly as it seems to me because they regard half of the program as mathematically unsound. Professor Williams indeed by a detailed analysis of several hypothetical cases attempts to prove the mathematics faulty. But his proofs, as I shall point out, are based on a misinterpretation of my paper. For he quotes me as *defining* a number infinity by the relation $k/0 = \infty$, where k cannot be zero.

To answer his argument, I must review a part of my original statement. I first recall briefly the service of infinite elements in modern geometry and, assuming the geometry as known, suggest that the advantages be carried over into algebra. I say that perhaps the best approach is through homogeneous equations but that we might begin by postulating an "improper number" ∞ which should play in algebra a rôle analogous to that played in geometry by the "improper point" at infinity on the line. The next few lines involve the heart of the matter at issue: "This number is now clothed with properties to conform to those of its geometric counterpart. Thus the x-intercept of the line

$$y = mx - k \tag{4}$$

is

$$x_0 = k/m. \tag{5}$$

Now if $m = 0$, the line is parallel to the x-axis and cuts it therefore at infinity. Accordingly, in virtue of (5) we attribute to ∞ the property

$$k/0 = \infty, \qquad k \neq 0, \tag{6}$$

for if $k = 0$ at the same time, the line (4) coincides with the x-axis and intersects it at every point."

I cannot see how Professors Fort and Williams find in this a proposal to define the symbol ∞ by the single relation (6).[4] For, attributing a property to a symbol

[1] This MONTHLY, September, 1922, p. 290.

[2] *Ibid.*, July-August, 1923, p. 255.

[3] *Ibid.*, November, 1923, p. 384.

[4] Professor Williams says that I propose "to create a number system in which there shall exist a single infinite number ∞ defined by the relation (6)" while Professor Fort states: "It

is quite a different thing from defining it. I gave in fact no formal definition at all. I only postulated the existence of the symbol ∞, attributed to it one property and stated a theorem about infinite roots of equations—for this was all I had occasion to use in the illustrations considered in my paper. Other properties than (6) are however implied by the context, especially by the phrase "*clothed with properties to conform to those of its geometric counterpart.*" Further I remark in a footnote that other properties may be assigned by considering the intercepts of the parabola $y = ax^2 + bx + c$.

Indeed relation (6) does not even say that k cannot be zero, a restriction that Professor Williams uses at least five times.[1] What it does say is that $k/0 = \infty$, *if* $k \neq 0$. If $k = 0$ the expression is indeterminate and might have any value including ∞, since the line (4) then coincides with the x-axis and meets it at each of its points, including the point at infinity. In other words $0 \cdot \infty$ is also indeterminate and might have the value 0, contrary to Professor Williams' assertion. This disposes of the first "contradiction" that arises when he seeks to show that $0 \cdot x + 0 = 0$ cannot have an infinite root.

Again when he finds by formal algebraic manipulation that there is doubt whether a certain line contains a point at infinity, a result in conflict with the geometry, what has he learned? Merely that the alleged "definition" (6) is inadequate. But why does Professor Williams ignore the qualification in my paper that the algebra *must conform to the geometry?* He might have found, I should think, quite as much mental recreation in ascertaining what additional property must be ascribed to the symbol ∞ in this case to make it fit the geometry as in speculating on what certain expressions mean in order to catch me in a logical inconsistency.

Since Professor Williams objects that my little sketch falls short of the treatment of number systems in books on the theory of functions, it will be instructive to see how an authority of his own choosing handles the question under discussion. Turning to Burkhardt,[2] § 12, I find this statement: "*In addition to the complex numbers and their symbols already introduced we introduce now a new one, 'infinity,' with the symbol* ∞, *which is to be regarded as the result of the division* 1/0." This he parallels with a geometrical definition of a point at infinity.[3] He then goes on to qualify the symbol by several other conventions and points out that $\infty \pm \infty$, $0 \cdot \infty$, and ∞/∞ as well as 0/0 are indeterminate forms. A few lines farther on he says: "According to conventions of this kind, certain words and symbols previously defined are assigned a wider meaning. That this procedure is permissible we have repeatedly stated in the first chapter; that it is useful is justified by results." Still later he reconciles this view with

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is with a kind of horror that I read where the author advocates the postulation of 'The Number Infinity' defined by (6)."

[1] In such statements as : "for $k \neq 0$, since by definition $k \neq 0$, we do not know the value of k except that it is not 0, which is untrue since $k \neq 0$ by hypothesis."

[2] *Theory of Functions of a Complex Variable*, Rasor Translation.

[3] He uses the convention of a point instead of a line at infinity in the plane, but the *principle* of introducing corresponding concepts into analysis and geometry is the same as I advocated.

the meaning of the symbol as used in the theory of limits and proves that: "*These two views of the symbol* ∞ *as applied to rational functions are not contradictory; and every such equation* $f(z_0) = \infty$, $f(\infty) = w_0$, $f(\infty) = \infty$ *which is true from the one point of view is also true from the other.*"

Now let Professor Williams turn to the strictly logical development of projective geometry by Veblen and Young [1] and he will find no vagueness about the symbol ∞ as used in my article. When establishing the non-homogeneous coordinate system in one dimension they say: "For the exceptional point P_∞, let us introduce a special symbol ∞ with exceptional properties, *which will be assigned to it as the need arises.* Some five or six properties, which are essentially the same as those of Burkhardt, are then assigned from the theory of collineations."

Professor Fort challenges the statement that my suggestions are in harmony with sound European tradition.[2] I trust that I have in a measure divested "my infinity" of its "horror" for Professor Fort and I must remind him that my remark *applied to the imaginary as well as the infinite.* Let me adduce briefly the evidence.

First I find this statement emanating from Göttingen which all but convicts me of what Sylvester once called unconscious plagiarism:[3] "These two equations [a linear and a quadratic homogeneous in three variables] have then precisely two common solutions [*Lösungstripel* $x : y : z$], *i.e., a curve of the first and one of the second order always intersect in exactly two points, which indeed may be real or complex, finite or infinite, distinct or coincident.*" [4]

Again in a textbook originating in Paris [5] and intended for pupils from fourteen to seventeen years old, after a very careful discussion of the equation $ax = b$ when $a = 0$, $b \neq 0$,—a discussion to which the most meticulous cannot take exception,—they summarize (p. 214) as follows: "We *say then that for* $a = 0$, x *is infinitely great and indicate this solution by the symbol* ∞." The authors do not consider imaginary points but neither do they use imaginary numbers in the solution of quadratics. They do however introduce the point at infinity on a line (p. 177) and assign to it a coördinate and they denote one point on the line by the symbol ∞— and this in a book on arithmetic and algebra.

O. Staude (*Analytische Geometrie*) introduces the point at infinity on a line as well as imaginary points, discusses infinite roots of a quadratic equation and assigns the label ∞ to one point of the line.

Chrystal [6] devotes ten pages to the interpretation of infinite solutions of equations. For the equation $ax + b = 0$, when $a = 0$, $b \neq 0$ he obtains an

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[1] *Projective Geometry*, vol. I, p. 54 ff. They note however that the symbol ∞ does not represent a number of a *field* as defined earlier. The Editor in a footnote to my paper called attention to this treatment but Professor Williams omits to mention it.

[2] He says: "I am sure in particular that his infinity was not in use in the teaching at Göttingen in 1912 or in Paris in 1913 when I was a student at those universities."

[3] See my paper, this MONTHLY (*1922*, 294).

[4] Klein, *Elementarmathematik vom höheren Standpunkte aus*, Teil II, p. 244.

[5] E. Borel-P. Stäckel, *Die Elemente der Mathematik*, I: *Arithmetik und Algebra.*

[6] *Algebra*, vol. I, Second Edition, pp. 385–395.

infinite root much as we assigned the property (6) to the symbol ∞. Again in treating the two linear equations

$$ax + by + c = 0, \qquad a'x + b'y + c' = 0$$

he obtains as the condition for parallelism of the lines $ab' - a'b = 0$ and proceeds: "We have thus fallen on the excepted case of §§ 4 and 5. If we assume that the results of the general formulæ obtained for the case $ab' - a'b \neq 0 \cdots$ hold also when $ab' - a'b = 0$, we see that in the present case neither of the numerators can vanish. · · · It follows then that

$$x = \frac{bc' - b'c}{0} = \infty, \qquad y = \frac{ca' - c'a}{0} = \infty;$$

and the analytical result agrees with the graphical."

These quotations are to be read like all other mathematical statements,—not excluding those occurring in discussion columns,—in the light of the context and applied as Chrystal warns "with a proper regard to accompanying circumstances."

But half of my proposal,—and not the lesser in importance,—concerned the imaginary in geometry. Instead of calling $x^2 + y^2 + 1 = 0$ an imaginary circle as I advocate, Professor Williams thinks it would be "more original and entertaining" to call it a pair of quaternion lines. Perhaps he is right—I had not considered the entertaining possibilities. Again he tells us that "to most people, of whom the writer is one, . . . it will be more interesting to confine themselves to points in the real plane, and they will not consider that they are telling only part of the truth when they say that $x^2 + y^2 = 0$ represents only a single point."

To many geometers the question of which elements of a curve or configuration are real and which imaginary is a matter of comparative indifference—the whole figure is their concern. Certain it is that no distinction is made in such fundamental formulas as Plücker's equations relating to the singularities of curves. Some of the most beautiful configurations in geometry, as for example that associated with the flexes of the plane cubic, are largely imaginary. And I regard it as a kind of hyper-esthetic sense that enables the mathematician to contemplate the intricate symmetry and beauty of a geometric figure which transcend the ordinary senses. I cannot profess to speak for most geometers, to say nothing of most mathematicians or most persons, but such geometers as Poncelet, Chasles, Darboux, Plücker, von Staudt, Lie and Study have concerned themselves with the imaginary in geometry so that one must conclude that they did not find it wholly devoid of interest. In particular, Salmon, Cayley, Clebsch, Klein, Pascal's *Repertorium* and the mathematical encyclopædias are on record as referring to a conic of the elliptic type and vanishing discriminant as a pair of imaginary lines. I am content to stand in such geometric company as this.

Finally, not to lose sight of the purpose of my original paper, let me recall the opening lines of Klein's *Higher Geometry*:[1] "We distinguish in general two

[1] *Einleitung in die höhere Geometrie*, vol. I, p. 1.

kinds of geometry, the synthetic geometry, which considers the figures themselves, and the analytic geometry which builds up its system essentially with the aid of analysis. Besides these two kinds of geometry, we may construct still a third kind which in a sense is the inverse of the two and which will form the subject of the present course of lectures. Thus whereas we ordinarily apply analysis to geometry, it shall be the purpose of this exposition inversely to apply geometry to analysis, to get acquainted with analytic relations in a geometric manner, or somewhat more precisely expressed, *with the aid of geometry to gain an insight into the theory of functions of several variables.*" If the dualism here indicated between algebra and geometry is to be without exception, we shall require on the one hand imaginary points in geometry to correspond to imaginary solutions of equations while on the other we shall need a symbol ∞ to correspond in the non-homogeneous coördinate system to the point at infinity on every line. In accordance with a cardinal principle of mathematical development, I advocated both steps to the end that algebra and geometry appear as merely different aspects of the same abstract truth. This concept of algebra and geometry as dualistic or isomorphic in my view leads to a more adequate picture of analytic geometry and is at the same time a distinct advantage in teaching.

II. Early History of Division by Zero.

By H. G. Romig, University of California.

In the development of algebra the question was bound to arise whether zero could be used in division. It was easily seen that $a + 0 = a$, $a - 0 = a$, and $a \cdot 0 = 0$, but the quotient resulting from the division by zero was questionable.

In 628 A.D. Brahmagupta [1] first speaks of the division by cipher, but gives no quotient. Bháscara,[2] in 1152, terms such a quotient infinite. John Wallis [3] first declares in 1657 that zero is no number but he introduces the form $1/0 = \infty$, being the first to use the symbol ∞ for infinity. He states further [4] that $1/3 < 1/2 < 1/1 < 1/0 < 1/(-1)$, etc., where he considers negative fractions greater than the infinite.

Isaac Newton [5] speaks of the quotient resulting from a division by zero, obtained by integrating dx/x, viz., $x^0/0$, as the infinite area under a hyperbola. In 1716 John Craig [6] declares that zero must be an infinitesimal for if absolutely of no value it cannot be used as a divisor, but his ideas are not altogether clear, for he repeats Wallis' mistake of considering negative fractions greater than

[1] Henry T. Colebrooke, *Algebra with Arithmetic and Mensuration from the Sanscrit of Brahmagupta and Bhâscara*, Chap. XVIII, Sec. II, "Algorithm," London, 1817, § 35–36, pp. 339–340.

[2] Henry T. Colebrooke, *idem*, Chap. II, Sec. IV, "Cipher," § 44–45, and footnote 5, p. 19.

[3] John Wallis, *Opera* I, London, 1695, Chap. IV, p. 27; Prop. LXIV, p. 395; Prop. CIV, p. 409.

[4] Moritz Cantor, *Vorlesungen über Geschichte der Mathematik*, Vol. II, Leipzig, 1913, p. 902.

[5] Isaac Newton, *Opuscula* (ed. Johann von Castillon), vol. I, Lausannæ and Genevæ, 1744, p. 4.

[6] George Cheyne, *Philosophical Principles of Religion*, Part II, London, 1716, pp. 167–169.

infinity. Bishop Berkeley[7] states that "zero is no number," while Landen[8] "calls it a *mere Blank* or Absolute Nothing." Maclaurin asserts that Leibnitz[9] "owns them [infinites and infinitesimals] to be no more than fictions."

The absolute exclusion of zero as a divisor was stated by Martin Ohm[10] of Berlin in 1828. He defines zero by stating that $\alpha - \alpha = 0$, and a/b as that number which multiplied by b gives a. "If a is not zero, but b is zero, then the quotient a/b has no meaning," for the quotient "multiplied by zero gives only zero and not a, as long as a is not zero."

In 1832 Wolfgang Bolyai de Bolya of Hungary stated that "1/0 is an impossible quantity"[11] but that "if z tends towards 0, then $1/z$ tends towards infinity" and $1/z = \infty$.[12] His conclusion results from his definition of zero as "$1 - 1 = 0$."[13]

De Morgan, in 1831, speaks[14] of $1/0 = \infty$ without objecting to this expression, but in 1864 he questions the division by zero and states[15] that the lack of proper symbols is to a great extent the cause of the various interpretations of 1/0. A "terminal order of infinity" exists which x approaches in $1/x$ in order to be 0^{-1}, which is the infinity of algebra, but if 0 "means the total absence of quantity" we cannot expect it to obey all the ordinary laws. His final conclusion[16] was that "the extreme infinite is 1/0." Apparently, he interprets here division of a finite number ($\neq 0$) by absolute zero as yielding a new form of infinity, the "extreme infinite." Seemingly in contradiction with this is his assertion:[17] "There is one conclusion. . . . I throw away entirely, as absolutely false; it is that parallel lines meet at an infinite distance."

In 1864 William Walton[18] of Trinity College, Cambridge, likewise approaches the modern conception regarding the use of zero as a divisor when he leaves an absolute blank in the denominator 1/ when considering "nonentity" or absolute zero, and uses the symbol 0 for an infinitesimal. Chief Justice Cockle of Australia declares[19] that the use of absolute zero in the denominator is impossible, which is the modern point of view, but he fails to define or explain his reasons for taking this position.

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[7] Florian Cajori, *A History of the Conceptions of Limits and Fluxions in Great Britain from Newton to Woodhouse*, Chicago, 1919, p. 218.

[8] Florian Cajori, *idem*, p. 222.

[9] Colin Maclaurin, *A Treatise on Fluxions*, vol. 1, Edinburgh, 1742, pp. 39–40.

[10] Martin Ohm, *Lehrbuch der niedern Analysis*, vol. 1, Berlin, 1828, pp. 110, 112, 132; *Der Geist der Differential- und Integral-Rechnung*, Erlangen, 1846, pp. 18, 76.

[11] Wolfgang Bolyai de Bolya, *Tentamen*, sec. ed., vol. 1, Budapest, 1897, § 32, p. 45.

[12] Wolfgang Bolyai de Bolya, *idem*, § 21, pp. 86, 91.

[13] Wolfgang Bolyai de Bolya, *idem*, "Adnotationes," p. 620.

[14] Augustus De Morgan, "Study and Difficulties of Mathematics," *Library of Useful Knowledge*, vol. I, London, 1836, pp. 41–42. (Art. dated Nov. 22, 1831.)

[15] Augustus De Morgan, "On Infinity and on the Sign of Equality," *Cambridge Philosophical Transactions*, vol. 11, 1864, pp. 145–189.

[16] Augustus De Morgan, *idem*, p. 188.

[17] Augustus De Morgan, *idem*, p. 176, footnote 2.

[18] William Walton, "On a Paradox in Definite Integration," *Quarterly Journal of Mathematics*, vol. VI, Sept., 1864, p. 322.

[19] Chief Justice Cockle, "On Limits," *Messenger of Mathematics*, vol. IV, 1868, p. 117. (Com. Jan. 1867 from Oakwal, Queensland, Australia.)

Rudolf Lipschitz [20] explains in 1877 that in dividing F/m by F_1/m_1, where $F_m \neq 0$, it is not permissible to let $F_1/m_1 = 0$, for the reason that there exists no fraction which, when multiplied by zero, gives F/m. The class concept was stated by Axel Harnack of Dresden in 1881, who used rational numbers in his calculus and from his definition of class he says [21] that the use of zero as a divisor is impossible. Four years later Stolz elaborates [22] upon this definition of number class and excludes zero as a divisor. The students of elementary mathematics accepted this exclusion principle and developed it in their textbooks using the definition of division but failed to use the definition of number class.

III. Dominical Letter and Perpetual Calendars.

By W. K. Nelson, University of Colorado.

The article in the November–December, 1922, issue of the Monthly on Uncle Zadock's rule for obtaining the dominical letter for any year by Mr. Vail was of interest to the writer and perhaps the following discussion of a similar rule and of two perpetual calendars will be of interest to others.

In Rietz and Crathorne's *College Algebra*, page 29, appears the following formula: [1]

$$D = \left\{ \frac{P + 2q + \left[\frac{3(q+1)}{5}\right] + N + \left[\frac{N}{4}\right] - \left[\frac{N}{100}\right] + \left[\frac{N}{400}\right] + 2}{7} \right\}_r,$$

where D is the day of the week counting Sunday the first, P is the day of the month, q is the number of the month in the year, counting January and February as the 13th and 14th months of the preceding year; and N the year. The brackets indicate that the largest integer contained is to be used and the braces with the subscript r indicate that the remainder is to be taken.

The dominical or Sunday letter for any common year is determined by 9 minus the day of the week upon which January 1st falls. By substituting in the above formula $P = 1$, $q = 13$, $N = 100C + T$, we obtain

$$L = 8 - \left\{ \frac{5C + \left[\frac{C}{4}\right] + T + \left[\frac{T}{4}\right]}{7} \right\}_r,$$

where L is the number indicating the dominical letter, C the centurial number and T the odd years. To avoid using the preceding year for the months of January

[20] Rudolf Lipschitz, *Lehrbuch der Analysis*, Bonn, 1877, pp. 27–28.

[21] Axel Harnack, *Die Elemente der Differential- und Integral-Rechnung*, Leipzig, 1881, p. 5; Eng. Ed. by Cathcart, 1891, p. 4.

[22] Otto Stolz, *Vorlesungen über Allgemeine Arithmetik nach den Neueren Ansichten*, Part I, Leipzig, 1885, p. 52.

[1] Quoted from C. Zeller, *Acta Mathematica*, vol. 9 (1887), pp. 131–6. Editor.

PERPETUAL CALENDAR

A

S	M	T	W	T	F	S
1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31				

B

S	M	T	W	T	F	S
						1
2	3	4	5	6	7	8
9	10	11	12	13	14	15
16	17	18	19	20	21	22
23	24	25	26	27	28	29
30	31					

C

S	M	T	W	T	F	S
					1	2
3	4	5	6	7	8	9
10	11	12	13	14	15	16
17	18	19	20	21	22	23
24	25	26	27	28	29	30
31						

D

S	M	T	W	T	F	S
				1	2	3
4	5	6	7	8	9	10
11	12	13	14	15	16	17
18	19	20	21	22	23	24
25	26	27	28	29	30	31

E

S	M	T	W	T	F	S
			1	2	3	4
5	6	7	8	9	10	11
12	13	14	15	16	17	18
19	20	21	22	23	24	25
26	27	28	29	30	31	

F

S	M	T	W	T	F	S
		1	2	3	4	5
6	7	8	9	10	11	12
13	14	15	16	17	18	19
20	21	22	23	24	25	26
27	28	29	30	31		

G

S	M	T	W	T	F	S
	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30	31			

	★	★	★	★	★	★	★
JULIAN	6	5	4	3	2	1	0
CALENDAR	13	12	11	10	9	8	7
				17	16	15	14
GREGORIAN		16	15		22		21
CALENDAR		20	19		18		17

DIRECTIONS: Find the hundreds part, or first two figures of year, in above table and last two figures of year in bottom table. Connect by a straight line the stars at the tops of the columns containing these figures. This line will cross a star at the head of a column in the middle table. The letter in this column opposite the required month indicates the proper calendar at the left of this page.

Use the first of the two letters opposite January and February if year is a common year and second if year is a leap year. Leap years are in Italics, and are marked +.

	No. days in Month	★	★	★	★	★	★	★	★	★	★	★	★
JAN.	31	BC	AB	GA	FG	EF	DE	CD	BC	AB	GA	FG	EF	DE
FEB.	28-29	FG	EF	DE	CD	BC	AB	GA	FG	EF	DE	CD	BC	AB
MAR.	31	F	E	D	C	B	A	G	F	E	D	C	B	A
APR.	30	C	B	A	G	F	E	D	C	B	A	G	F	E
MAY	31	A	G	F	E	D	C	B	A	G	F	E	D	C
JUN.	30	E	D	C	B	A	G	F	E	D	C	B	A	G
JUL.	31	C	B	A	G	F	E	D	C	B	A	G	F	E
AUG.	31	G	F	E	D	C	B	A	G	F	E	D	C	B
SEPT.	30	D	C	B	A	G	F	E	D	C	B	A	G	F
OCT.	31	B	A	G	F	E	D	C	B	A	G	F	E	D
NOV.	30	F	E	D	C	B	A	G	F	E	D	C	B	A
DEC.	31	D	C	B	A	G	F	E	D	C	B	A	G	F

The letters of the top row are dominical or Sunday letters. The letter to the left is to be used, except for the first two months of leap year, when the second one is used.

★	★	★	★	★	★	★
00	01	02	03		*04+*	05
06	07		*08+*	09	10	11
	12+	13	14	15		*16+*
17	18	19		*20+*	21	22
23		*24+*	25	26	27	
28+	29	30	31		*32+*	33
34	35		*36+*	37	38	39
	40+	41	42	43		*44+*
45	46	47		*48+*	49	50
51		*52+*	53	54	55	
56+	57	58	59		*60+*	61
62	63		*64+*	65	66	67
	68+	69	70	71		*72+*
73	74	75		*76+*	77	78
79		*80+*	81	82	83	
84+	85	86	87		*88+*	89
90	91		*92+*	93	94	95
	96+	97	98	99		

The years 1700, 1800, 1900, 2100, 2200 in Gregorian Calendar are not leap years.

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1924

EXAMPLE

Find day of week of Armistice Day, Nov. 11, 1918. The line drawn determines the calendar for 1918, November's calendar being C. Then Nov. 11th falls on Monday.

FIG. 1.

and February as suggested by the original formula, the function of q for these two months was decreased by one. This makes it necessary to decrease the day of the week by one and increase the dominical letter by one for January and February of a leap year. The above rule stated in words is probably easier to remember than Uncle Zadock's rule: *To five and one fourth times the centurial number, dropping fractions, add one and one fourth times the odd years, dropping fractions.*

Divide this sum by 7 *and subtract the remainder from* 8.

This last remainder indicates the dominical or Sunday letter for the year.

It is a simple matter to construct a nomograph of the above equation involving L, C, and T. Let

$$X = 8 - \left\{\frac{5C + \left[\frac{C}{4}\right]}{7}\right\}_r, \qquad Y = \left\{\frac{T + \left[\frac{T}{4}\right]}{7}\right\}_r.$$

Then $L = X - Y$. Parallel scales may be plotted for X and Y with the same modulus, then following Lipka's method of the "Three Parallel Scales Chart," the scale for L will be half way between the other scales and have a modulus half the size. Then if a line is drawn from a point on the first scale determined by the first two figures of the year to a point on the third scale indicated by the last two figures of the year, it will cross the middle scale at a point which indicates the dominical letter for the year.

Since the calendar for any year is determined by the dominical letter for that year, it is possible to add to the nomograph and construct a rather compact perpetual calendar as shown in Fig. 1. The first letter in each column opposite January is the dominical letter for a common year and the last ten months of a leap year, the second letter being the dominical letter for January and February of a leap year.

By a method similar to that described the nomograph was extended to include Julian dates as well as the Gregorian. Thus a calendar for twenty-three hundred years is obtained, that is, from the year 1 to the end of 2299.

A very complete and simple perpetual calendar may be made by using the principle of the circular slide rule. The slide rule is a mechanical means of adding numbers. In the ordinary slide rule, logarithms of numbers are added. Here the same principle is used to add functions of C, T, and q to get a function of the day of the week. D, the day of the week for the first day of the month, may be expressed as a function of C, T, and q, by replacing in the first equation P by 1 and N by $(100C + T)$.

Take two such discs as the one shown in Fig. 2 and cut one of them along the broken lines so that we have one large disc and one smaller one with seven openings in it. Fasten the discs together at the center, so that they are free to turn, and a calendar will be obtained which gives the calendar for any year from the year one to the year 2099. To get the calendar for 1923 turn the smaller disc until the 23 is one of the group of numbers in the opening adjacent to 1900, then

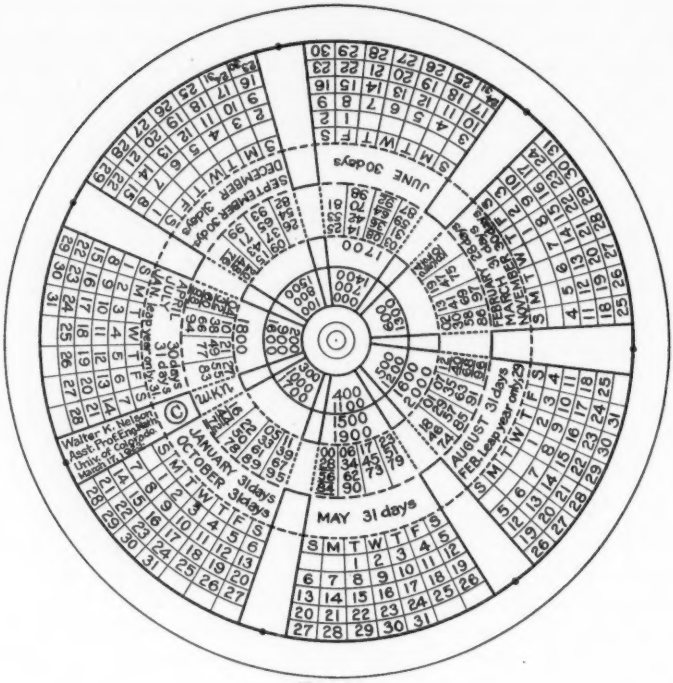

Fig. 2.

the names of the months are opposite the correct monthly calendars. (Fig. 2 shows the discs in this position.) This setting also shows all the years since the beginning of the Christian era which had this same calendar. The numbers in the inner ring are centurial numbers of the Julian calendar, while centurial numbers of the Gregorian calendar are in the adjacent ring.

IV. Some Results Involving π.

By R. S. Underwood, Alabama Polytechnic Institute.

Let

$$f_0(x) = \log (a + x).$$

$$f_1(x) = \int_0^x f_0(x)dx,$$

$$f_2(x) = \int_0^x f_1(x)dx,$$

and generally

$$f_n(x) = \int_0^x f_{n-1}(x)dx,$$

where a is any number, real or complex, except one that is negative or zero, and where the principal logarithm is used, and the integrations are made over the real interval $(0, x)$.

By integration,

$$f_1(x) = (x + a)[\log (x + a) - 1] - a \log a + a,$$

and finally, as may be shown by induction,

$$n!\, f_n(x) = (x + a)^n[\log (x + a) - s_n] + \log a[x^n - (x + a)^n] + \sum_{r=1}^{n} s^r \binom{n}{r} a^r x^{n-r}, \quad (1)$$

where $s_r = 1 + 1/2 + 1/3 + \cdots + 1/r$.

Suppose further that

$$g_0(x) = \tan^{-1}x, \qquad g_n(x) = \int_0^x g_{n-1}(x)dx.$$

Now $\tan^{-1} x = \pi/2 - (i/2) \log (x - i) + (i/2) \log (x + i)$.

Hence, by applying (1), and making some reductions,[1]

$$n!\, g_n(x) = \left\{ x^n - \binom{n}{2} x^{n-2} + \binom{n}{4} x^{n-4} - \cdots \right\} \tan^{-1} x - \left\{ \binom{n}{1} x^{n-1} - \binom{n}{3} x^{n-3} + \cdots \right\} \{\log \sqrt{(1 + x^2)} - s_n\} - \left\{ \binom{n}{1} s_1 x^{n-1} - \binom{n}{3} s_3 x^{n-3} + \cdots \right\}. \quad (2)$$

Hence, on setting $x = 1$, and using Demoivre's Theorem,

$$n!\, g_n(1) = 2^{n/2}\{\cos (n\pi/4)\}\pi/4 - 2^{n/2}\{\sin (n\pi/4)\} \left(\tfrac{1}{2} \log 2 - s_n \right) - \left\{ \binom{n}{1} s_1 - \binom{n}{3} s_3 + \cdots \right\}. \quad (3)$$

Again, starting from

$$\tan^{-1} x = x - x^3/3 + x^5/5 - \cdots, \quad (-1 < x \leqq 1),$$

we get (using term by term integration of uniformly convergent series)

$$g_n(1) = 0!/(n + 1)! - 2!/(n + 3)! + 4!/(n + 5)! - \cdots. \quad (4)$$

On replacing n by $4n$ and $4n - 2$ respectively in (3), and comparing with (4), we get

[1] Formula (2) may also be established directly by induction, without the use of either logarithms or imaginaries. EDITOR.

$$\pi = M + (-1)^n(4n)!\,2^{-2n+2}\{0!/(4n+1)! - 2!/(4n+3)! + \cdots\}, \quad (5)$$

$$\log 2 = N + (-1)^n(4n-2)!\,2^{-2n+2}\{0!/(4n-1)! - 2!/(4n+1)! + \cdots\}, \quad (6)$$

where

$$M = (-1)^n 2^{-2n+2}\left\{\binom{4n}{1}s_1 - \binom{4n}{3}s_3 + \cdots \pm 4n s_{4n-1}\right\}$$

and

$$N = 2s_{4n-2} + (-1)^n 2^{-2n+2}\left\{\binom{4n-2}{1}s_1 - \binom{4n-2}{3}s_3 + \cdots \pm (4n-2)s_{4n-3}\right\};$$

and hence both M and N are rational.

Also

$$\pi = \lim_{n\to\infty} M \text{ and } \log 2 = \lim_{n\to\infty} N.$$

From (5) we get the result that the sum of the infinite series

$$0!/(4n+1)! - 2!/(4n+3)! + \cdots$$

is a transcendental number for every positive integer n.

Specific series obtained by this method are

$$\pi = 0 + 4[0!/1! - 2!/3! + \cdots],$$

$$\pi/2 - \log 2 = 0 + 2[0!/2! - 2!/4! + \cdots],$$

$$\log 2 = 1 - 2[0!/3! - 2!/5! + \cdots],$$

$$\pi/2 + \log 2 = 5/2 - 3![0!/4! - 2!/6! + \cdots],$$

$$\pi = 10/3 - 4![0!/5! - \cdots],$$

$$\log 2 = \frac{79}{120} + 6!/2^2[0!/7! - \cdots],$$

$$\pi = 109/35 + 8!/2^2[0!/9! - \cdots],$$

$$\pi = 87217/27720 - 12!/2^4[0!/13! - \cdots].$$

RECENT PUBLICATIONS.

EDITED BY D. C. GILLESPIE, Cornell University, Ithaca, N. Y., to whom communications should be sent.

REVIEWS.

Relativity, a Systematic Treatment of Einstein's Theory. By J. RICE. London, Longmans, Green & Co., 1923. 389 pages.

Of works written originally in English and with the purpose of giving a *detailed treatment* of the general theory of relativity with *requisite proofs* this is, so far as the reviewer is aware, the third. The other two works are those of Eddington [1] and Birkhoff.[2] We do not hesitate to recommend the work of Professor Rice in preference to these two, to any reader who is but slightly acquainted with the general theory or is not well versed in the higher parts of mathematics *and* mathematical physics. In fact, our opinion of this book is so high that the only other work of its kind in any language that we would care to place with it is the *Relativitätstheorie* of M. v. Laue [3] which we believe has not been done into English.

Professor Rice's book is not large. Its 389 pages of text fall roughly into three parts. After an interesting introduction of 30 pages, the author devotes 130 pages to special relativity and 130 more to the general theory. The remainder of the book deals with world geometry.

The Einstein theory is a composite of physics and four-dimensional non-Euclidean geometry, and seldom is one well versed in both these fields. To the physicist the mathematics is very difficult and abstruse, to the mathematician the finer physical theories are all but incomprehensible. The task of any author who does not appeal to the trained specialist is thus one of extreme difficulty. We have only praise for the way Professor Rice has developed his material.

Few readers will find difficulty in the first part of the book which, as stated, treats of the restricted theory. The author makes haste slowly and the reader has time to orient himself in the new world. The author has perhaps given too much attention to the four-dimensional vector analysis; to some it may seem that he has missed the opportunity of introducing the reader to the tensor analysis by replacing these vectors by tensors based on the metric of the restricted relativity theory.

Leaving the restricted relativity we enter in Part II into the real gist of the matter and here difficulties abound in plenty. The author's first task is to develop the tensor theory based on the general quadratic form $g_{ij}dx_i dx_j$. This involves the introduction of Christoffel's symbols [4] $[\alpha\beta, \gamma]$ and $\{\alpha\beta, \gamma\}$, covariant differentiation, the Riemannian curvature tensor $R_{ijk}{}^l$ and the equations of a geodesic. The notion of parallel displacement has been deferred till a late

[1] A. S. Eddington, *Mathematical Theory of Relativity*, Cambridge University Press, 1923.
[2] G. D. Birkhoff, *Relativity and Modern Physics*, Harvard University Press, 1923.
[3] Two volumes, Braunschweig, Vieweg und Sohn, 1921.
[4] See L. P. Eisenhart, *Differential Geometry*, Boston, Ginn & Co., 1909.

chapter and the reader may well wonder how one ever arrived at the complicated expressions defining the $R_{ijk}{}^{l}$ and why these tensors play such a dominant part in Einstein's theory. Einstein's thesis is this:—the presence of gravitating matter affects the geometry of space. Denoting the momentum energy tensor by $T_{ij} = \rho(dx_i/ds)(dx_j/ds)$, he assumes the geometrical effect is proportional to T_{ij}. The most natural geometrical tensor to take is the contracted curvature tensor R_{ij}. This is not allowable because the conservation of energy and momentum requires the divergence of T_{ij} to be 0. As the divergence of R_{ij} is not 0 while that of $R_{ij} - \frac{1}{2}g_{ij}R$ is, Einstein takes

$$R_{ij} - \tfrac{1}{2}g_{ij}R = -uT_{ij}, \tag{1}$$

where R is the invariant curvature tensor obtained by contracting R_{ij}. Writing these equations in the equivalent form

$$R_{ij} = -u(T_{ij} - \tfrac{1}{2}g_{ij}T),$$

we see that in space devoid of matter $R_{ij} = 0$. The equation div $T_{ij} = 0$ leads at once to the geodesics

$$\ddot{x}_\lambda + \{\alpha\beta, \lambda\}\dot{x}_\alpha\dot{x}_\beta = 0$$

as the path of a free particle in the field of gravitation.

In chapter XIV we have a simple treatment of the three problems that have excited endless discussion in the newspapers and vain attempts to make the man in the street understand relativity. Alas there is no royal road to relativity. We refer of course to the baffling motion of the perihelion of Mercury (43″ per century), the prediction of the deflection of a ray of light in passing a strong gravitating body (eclipse phenomenon), and the shift of the spectral lines toward the red. The last is still a matter of controversy and investigation.

Part III is devoted to world geometry and can be appreciated only by the advanced student. Einstein's spherical cylindrical world is studied in detail and some account of de Sitter's spherical space-time is given. The reviewer would have welcomed a fuller account of the latter. It not only explains the systematic shift of the spectral lines of distant stars towards the red but as Silberstein has recently shown [1] it also affords a means of determining the size of the universe by measuring the parallax and radial velocities of stellar clusters. The result gives a value of $R = 6.10^{12}$ astronomical units, roughly, and so agrees well with the result obtained by Einstein based on Kapteyn's estimate of the number of suns in a cube of 10 parsecs about our sun.

On page 348 we read "Of course in elliptical space even with Einstein's hypothesis of cylindrical time, the existence of anti-suns, etc., would no longer follow." This, we believe, rests on a misapprehension. The time it would take is πR divided by the velocity of light. In de Sitter's theory [2] the velocity slows down to 0 as $r \doteq R$, but this is not so in Einstein's theory.

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[1] *Nature*, 1924, pp. 350 and 602.

[2] W. de Sitter, On Einstein's theory of gravitation, *Monthly Notices, Royal Astronomical Society*, vol. 78 (1916–17), p. 3.

The last chapter is devoted to the theory of Weyl [1] and its further essential development by Eddington.[2] The electromagnetic theory of Einstein is made to depend upon the contravariant electromagnetic potential tensor and there is no organic connection with the metric $g_{ij}dx_idx_j$ and the gravitational field. Weyl extends Riemann's metric by introducing a gauge system, depending on four quantities which he identifies with the four components φ^k. He effects this by generalizing the notion of parallel displacement by setting

$$\delta A^r = -\Gamma_{ik}{}^r A^i dx_k, \qquad \Gamma_{ik}{}^r = \Gamma_{ki}{}^r.$$

Here the Γ's are no longer the Christoffel symbols $\{ik, r\}$ but unknown quantities related to the φ^k and the g_{ij} by

$$\Gamma_{ik}{}^r = \{ik, r\} + \tfrac{1}{2}(g_i{}^s\varphi_k + g_k{}^s\varphi_k - g_{ik}\varphi^r).$$

The length of the vector A^r does not remain unchanged when carried around a closed circuit as defined by this relation, but only the ratio of two such vectors issuing from the same point. In this way Weyl reaches a geometric theory not only of the gravitational field but also of the electro-magnetic field as well.

In Eddington's theory a parallel displacement about a closed circuit leads to a generalized Riemannian curvature tensor

$$*R_{k,\ lm}{}^i = +\frac{\partial \Gamma_{kl}{}^i}{\partial x_m} - \frac{\partial \Gamma_{km}{}^i}{\partial x_l} + \Gamma_{rl}{}^i\Gamma_{km}{}^i - \Gamma_{rm}{}^i\Gamma_{kl}{}^r$$

whose contraction leads to $*R_{kl}$. The components may be broken into a symmetric and a skew symmetric tensor

$$*R_{kl} = g_{kl} + \varphi_{kl},$$

$$\varphi_{kl} = \frac{1}{2}\left(\frac{\partial \Gamma_{k\alpha}{}^\alpha}{\partial x_l} - \frac{\partial \Gamma_{l\alpha}{}^\alpha}{\partial x_k}\right).$$

As the ϕ's satisfy the familiar relation

$$\frac{\partial \phi_{kl}}{\partial x_m} + \frac{\partial \varphi_{lm}}{\partial x_k} + \frac{\partial \phi_{mk}}{\partial x_l} = 0,$$

we may regard the φ_{kl} as defining the electromagnetic tensor of Einstein's earlier theory, while the metric of our space is defined by $g_{kl}dx_kdx_l$. The book closes with an account of Einstein's recent paper [3] in which he endeavors to determine the 40 coefficients $\Gamma_{ik}{}^r$ by applying Hamilton's principle:

$$\delta \int H dt = 0,$$

where

$$H = 2\sqrt{-|R_{kl}|},$$

[1] H. Weyl, *Raum, Zeit, Materie*, Berlin, J. Springer, 1918.
[2] *Proceedings of the Royal Society (London)*, 1921, p. 104. Also his book referred to above.
[3] *Berliner Berichte*, 1923, p. 32.

$|R_{kl}|$ being the determinant formed of the R_{kl}. We quote the closing words of Einstein's paper: "The foregoing investigation shows that Eddington's general idea in connection with Hamilton's principle leads to a theory which is almost free from arbitrary hypotheses, which is in accord with our knowledge of gravitation and electricity and which unites these two theories in a really complete manner,"—a graceful tribute indeed to the great English astronomer.

Descartes urged that a novel theory should be made so transcendentally clear that the man in the street could understand it. We fear that the mythical man in the street must still be left in darkness but we believe that what is humanly possible to make the obscure clear, our author has achieved.

JAMES PIERPONT.

NOTES ON RECENT PUBLICATIONS.

The library of the Mathematical Association has received from P. WIJDENES, Amsterdam, the editor of *Nieuw Tijdschrift voor Wiskunde*, the following texts published by P. Noordhoff, Groningen: P. Wijdenes, *Nieuwe School-Algebra* (Deel I, II, III), and *Grafieken-Schrift;* Molenbroek and Wijdenes, *Planimetrie* (Deel I, II); Wijdenes, *Vraagstrukken over Hoogere Algebra en Rekenkunde;* Wijdenes, *Tien Jaargangen van het Nieuw Tijd. v. Wisk.* (Deel II); Versluys, *Tafel H* (logarithms); Molenbroek, *Leerboek der Stereometrie.* These examples of current Dutch texts, like other books in the Association library, are available to the members of the Association for inspection, on application to the Secretary.

ARTICLES IN CURRENT PERIODICALS.

The lists appearing regularly in the MONTHLY of articles in current periodicals are intended to include (1) titles of papers in all mathematical journals published in the United States; (2) titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.

AMERICAN JOURNAL OF MATHEMATICS, volume 46, no. 2, April, 1924: "Two-dimensional tensor analysis without coördinates" by G. Y. Rainich, 71–94; "Functional operations as applied to a class of Volterra integral equations" by H. T. Davis, 95–109; "Representation of three-element algebras" by B. A. Bernstein, 110–116; "The Riemann adjoints of completely integrable systems of partial differential equations" by C. A. Nelson, 117–130; "Further types of involutorial transformations which leave each cubic surface of a web invariant" by V. Snyder, 131–140.

ANNALS OF MATHEMATICS, second series, volume 25, no. 1, September, 1923: "The history of notations of the calculus" by F. Cajori, 1–46; "New applications of a fundamental theorem of substitution groups" by G. A. Miller, 47–52; "Geodesic representation between Riemann spaces" by H. Levy and A. Bramley, 53–56; "On the residues of figurate numbers" by O. E. Glenn, 57–70; "On symmetric forms in N variables. II" by A. Dresden, 71–84; "On an infinite system of non-abelian groups of order nm^{n-1}" by W. E. Edington, 85–90.

BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, volume 30, nos. 3–4, March–April, 1924: "Problems in involutorial transformations in space" by V. Snyder, 101–124; "A generalization of the syllogism" by B. A. Bernstein, 125–127; "On certain quinary quadratic forms" by E. T. Bell, 127–130; "The invariants of forms under the binary linear homogeneous group G_6 modulo 2" by O. E. Glenn, 131–139; "On the application of the theory of ideals to Diophantine analysis" by G. E. Wahlin, 140–154.

JOURNAL OF MATHEMATICS AND PHYSICS, Massachusetts Institute of Technology, volume 3, no. 4, May, 1924: "On dynamic stresses in pseudo-continuous media" by P. Heymans, 237–252.

SCHOOL SCIENCE AND MATHEMATICS, volume 24, no. 6, June, 1924: "Checking the results of classification in nine first year algebra classes by means of the Holz algebra scales" by E. W. Schreiber, 614–622; "Mistakes in the computation of standard deviations" by W. C. Eells, 623–626.

THE SCIENTIFIC MONTHLY, volume 18, no. 6, June, 1924: "The origin, nature and influence of relativity" by G. D. Birkhoff, 616–624.

UNDERGRADUATE MATHEMATICS CLUBS.

All reports of club activities should be sent to H. J. ETTLINGER, 2910 Harris Park Ave., Austin, Texas.

CLUB ACTIVITIES.

THE PASCAL CIRCLE, TRINITY COLLEGE, Washington, D. C.

[*1922*, 419.]

The officers for the year 1922–1923 were: honorary president, Professor Marie Cecilia Mangold; president, Margaret Kelly '23; vice-president, Margaret McAuliffe '23; secretary, Agnes Perrot '24; treasurer, Anne Foley '25.

The officers for the year 1923–1924 were: honorary president, Professor Marie Cecilia Mangold; president, Helen Keller '24; vice-president, Marguerite Dwyer '24; secretary, Helena Crowley '25; treasurer, Orillia Hollis '26.

The programs for the years 1922–1923 and 1923–1924 were the following:

October 23, 1922. Business meeting. It was decided that membership should be limited to those who could maintain an average grade of 80 per cent. in mathematics.

November 6, 1922. Discussion.

November 13, 1922. "The interest of mathematics" by Margaret Kelly '23. Geometrical fallacies.

December 6, 1922. Annual social.

February 26, 1923. Geometrical fallacies.

May 23, 1923. Election for the year 1923–1924.

October 23, 1923. "History of the Pascal Circle" by Helen Keller '24. "Life of Pascal" by Helen McMahon '24.

November 13, 1923. "Mathematics and psychology" by Agnes Perrot '24. Article on the seismograph from the *Scientific Monthly* by Elizabeth Frank '26.

November 18, 1923. "The seismograph" by Rev. Francis Tondorff, S.J., Director of the Seismological Laboratory of Washington, D. C.

December 18, 1924. "Women in mathematics" by Ruth Eileen Lynch '26. Problem presented by Helena Crowley '25.

February 3, 1924. An account of the prize award to a Chicago mathematician for work in science by Helen Keller '24. "The history of mathematics" by Helen McMahon '24. Problem presented by Blanche Brunini '25.

March 26, 1924. Discussion on "The value of mathematics in training the mind" introduced by Helen Sheehan. Trick with cards by Rose O'Donnell '25.

May meeting. Election for the ensuing year.

(Report by Professor Mangold.)

THE MATHEMATICS CLUB OF HOOD COLLEGE, Frederick, Md.

The club had a membership of twenty-five during the year 1923–1924. The meetings were held monthly. The following were the officers for the year: president, Helen Goodfellow '24;

secretary-treasurer, Irene Sauserman '25; reporter, Cordelia Brong '24. The following topics were presented at the meetings:

November 5, 1923. "The cattle problem of Archimedes" by Ruth Feaga '24. Huge numbers and their multiplication by Norma Swayne '24.

December 3, 1923. Magazine evening. "Mathematical problems in relation to the history of economics and commerce" by Regina Beachley '25. "Mathematical concepts in current literature" by Cordelia Brong '24. "Values in high school mathematics" by Ruth Michael '24.

February 4, 1924. "Mathematical treatment of the data of science" by Miss Nelle Ingels of Washington, D. C.

March 3, 1924. "The slide rule, its history and application" by Flora Morton '25, Ruth Reiver '25, Mary H. Moomau '25 and Irene Sauserman '25.

April 7, 1924. "The triangle and its circles" by Helen Rodgers '24.

May 5, 1924. "Basic geometrical constructions found in designs of tiling, parquet flooring and Gothic tracery" by Mary Neighbors '24, Hilda Rickard '25 and Grace Brown '25.

May 17, 1924. Picnic for all the members.

(Report by Miss Brown.)

The Tulane Mathematical Society of Tulane University, New Orleans, La.

The Tulane Mathematical Society was organized in 1921 for the purpose of providing a field of activities for the advanced students at Tulane, and the teachers of mathematics in New Orleans. Since its organization it has met three times a year. The papers have been on historical or pedagogical subjects, or of an elementary research nature such as Master's theses. The club was fortunate in securing two nationally known mathematical lecturers. The program was as follows:

Fall meeting: "The areas of certain types of cones" by Miss W. D. Daly of the Sophie B. Wright High School.

Winter meeting: "Romance of the number system" by Professor H. E. Slaught of the University of Chicago.

Spring meeting: "Algebras and their arithmetics" by Professor L. E. Dickson of the University of Chicago.

(Report by Professor Buchanan.)

Pi Mu Epsilon of Iowa State College of Agriculture and Mechanic Arts, Ames, Iowa.

During the year 1922–1923 the following programs were given by the mathematics club:

October 18, 1922. Business meeting at which the following officers were elected: president, H. P. Doole, Gr.; secretary, H. A. Hoover, Gr.

November 8, 1922. About one hundred persons were present to hear a discussion of "The theory of relativity" by Professor J. S. Turner and Dr. E. S. Allen.

December 6, 1922. Mr. J. V. Longnecker '23 and Mr. M. E. Nordberg '24 gave an interesting explanation of the theory and practice of the slide rule. More than one hundred and twenty-five were present at this meeting. At the business meeting following the program Mr. F. A. Gardner, Gr., was elected vice-president of the club and Miss Marion Miller '24 was elected treasurer.

January 10, 1923. "Determinants and their applications" by Mr. Brandner, Gr.

February 7, 1923. "Logarithmic and exponential curves" by Miss Marion Miller '24 and Mr. Arthur Eschebach, Gr.

March 7, 1923. "The fourth dimension" by Professor G. W. Snedecor.

April 4, 1923. "The theory of error" by Mr. H. P. Doole, Gr. Professor Daniells reported that the national fraternity, Pi Mu Epsilon, has taken favorable action on the petition of the club for a chapter of the organization.

April 25, 1923. The installation of the chapter of Pi Mu Epsilon took place at the home of Professor Marian E. Daniells and seventeen members were initiated into the fraternity. Dr. E. R. Smith spoke on the principles of the fraternity and an amusing mathematical program was given.

May 2, 1924. "Indeterminate forms" by Miss Helen Nugent '23. "Equations of wave motion" by Mr. H. A. Hoover, Gr. Election of officers for the year 1923–1924 resulted as follows: director, Mr. A. H. Eschebach, Gr.; vice-director, Mr. H. P. Doole, Gr.; secretary, Mr. M. E. Nordberg '24; treasurer, Miss Marion Miller '24; librarian, Mr. William Nielsen '25.

November 5, 1924. Business meeting. Director Eschebach presented his resignation and Mr. Doole was elected to the office, Mr. Roger I. Wilkinson being made vice-director.

November 27, 1924. Initiation and banquet. Vice-director Wilkinson gave a toast of welcome to which Miss Ruth Dewey responded. Dr. E. R. Smith spoke of the history and advantages of honorary fraternities.

December 13, 1924. Dr. Turner gave an interesting talk on "Early texts and manuscripts of mathematics" and showed his own collection of old books.

January 11, 1924. "The planimeter" by Mr. Roger I. Wilkinson '24. "The game of Nim" by Mr. W. Lee Harris, Gr. Report of the national convention of Pi Mu Epsilon by Dr. E. R. Smith.

February 1, 1924. Business meeting.

February 14, 1924. "Conic sections" by Dr. E. S. Allen

April 10, 1924. After the initiation of six new members, the officers for the coming year were elected: director, Professor Marian E. Daniells; vice-director, Mr. W. Lee Harris, Gr.; secretary, Miss Leora Porter '25; Miss Helen Smith, instructor.

On the second of May the fraternity gave a dinner for the visiting mathematicians in attendance at the annual meeting of the Iowa Section of the Mathematical Association of America. The following program of toasts was given:

Hall of Mathematics.

Guide	Dr. E. R. Smith,
Lower Floors	Mr. H. C. Tingleff '24,
Recreation Room	Pi Mu Epsilon Sax-tet,
Upper Floor	Dr. H. L. Rietz, University of Iowa,
Rotunda	Dean Maria M. Roberts,
Roof	Professor F. M. McGaw, Cornell College.

(Report by Miss Leora Porter, Secretary.)

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.]

3095. Proposed by GEORGE RUTLEDGE, Massachusetts Institute of Technology.

Establish the following n identities involving the binomial coefficients of any even order:

$$1!\left\{\frac{1}{1}\binom{1}{1}\binom{2n}{n-1}-\frac{1}{2}\binom{2}{1}\binom{2n}{n-2}+\frac{1}{3}\binom{3}{1}\binom{2n}{n-3}-\cdots+(-1)^{n-1}\frac{1}{n}\binom{n}{1}\binom{2n}{0}\right\}=\frac{1}{2}\binom{2n}{n}.$$

$$3!\left\{\frac{1}{2}\binom{3}{3}\binom{2n}{n-2}\sum_{\binom{n-1}{1}}^{2}-\frac{1}{3}\binom{4}{3}\binom{2n}{n-3}\sum_{\binom{n-1}{1}}^{3}+\cdots+(-1)^{n-2}\frac{1}{n}\binom{n+1}{3}\right.$$
$$\left.\times\binom{2n}{0}\sum_{\binom{n-1}{1}}^{n}\right\}=\frac{1}{2}\binom{2n}{n}.$$

$$5!\left\{\frac{1}{3}\binom{5}{5}\binom{2n}{n-3}\overset{3}{\Sigma}_{\binom{n-1}{2}}+\cdots(-1)^{n-3}\frac{1}{n}\binom{n+2}{5}\binom{2n}{0}\overset{(n)}{\Sigma}_{\binom{n-1}{2}}\right\}=\frac{1}{2}\binom{2n}{n}.$$

. . . . . .

$$(2n-1)!\left\{\frac{1}{n}\binom{2n-1}{2n-1}\binom{2n}{0}\overset{(n)}{\Sigma}_{\binom{n-1}{n-1}}\right\}=\frac{1}{2}\binom{2n}{n}.$$

The notation $\overset{(j)}{\Sigma}_{\binom{n-1}{i-1}}$ is used to indicate the sum of the $\binom{n-1}{i-1}$ products of the squared reciprocals of the first n integers excepting j, taken $i-1$ at a time.

The first of these identities is well known in the form

$$\binom{2n}{0}-\binom{2n}{1}+\binom{2n}{2}-\cdots+\binom{2n}{2n}=0$$

and the last one is obvious.

3096. Proposed by W. J. SIDIS, New York City.

In a scale of numeration whose radix is prime,

(1) There cannot be four distinct digits whose cubes all end in the same digit.

(2) If the cubes of two distinct digits end alike (that is, in the same digit), there will always be a third such digit.

(3) Under the conditions of (2), given any digit except 0, there will be two digits whose cubes have the same last figure as the cube of the given digit.

An extension of (2) and (3) is possible. If n is an odd prime, then, if the nth powers of two distinct digits end alike, there will be a group of n distinct digits whose nth powers end alike (that is, in the same figure), and such a group may be made to include any given digit not 0.

As an extension of (1), we may say that, under the original conditions, there cannot be $n+1$ digits whose nth powers end alike.

3097. Proposed by A. A. BENNETT, University of Texas.

Show that there are five distinct types of sets admitting associative multiplication and containing but two elements. Determine the number of types of sets admitting associative multiplication and containing three elements.

SOLUTIONS.

3044 [1923, 449]. Proposed by EUGENE M. BERRY, West Lafayette, Indiana.

1. Express $\prod_{k=1}^{k=n}\cos a_k$ as a summation of cosines, each term to be of the form $C\cos(\pm a_1 \pm a_2 \pm a_3 \pm \cdots \pm a_n)$.

2. Express $\prod_{k=1}^{k=n}\sin a_k$ as a summation of sines or cosines according as n is odd or even.

3. Express $\prod_{k=1}^{k=n}\sin a_k \cdot \prod_{j=1}^{j=m}\cos b_j$ as a summation of sines or cosines according as n is odd or even.

SOLUTION BY THE PROPOSER.

We will take the third one first as the first two are special cases of it.

$$2^{n+m}i^n\prod_{k=1}^{n}\sin a_k\cdot\prod_{j=1}^{m}\cos b_j=2^{n+m}i^n\prod_{k=1}^{n}\frac{e^{ia_k}-e^{-ia_k}}{2i}\cdot\prod_{j=1}^{m}\frac{e^{ib_j}+e^{-ib_j}}{2}$$
$$=\prod_{k=1}^{n}(e^{ia_k}-e^{-ia_k})\cdot\prod_{j=1}^{m}(e^{ib_j}+e^{-ib_j}). \qquad (1)$$

Expressed as a sum of exponential terms any term t will be of the form $\pm e^{i(\pm a_1\pm a_2\pm\cdots\pm a_n\pm b_1\pm b_2\pm\cdots\pm b_m)}$. Since in (1) e^{-ia_k} carries with it a negative sign before the e, the sign before any term will be $+$

$+$ or $-$ according as the number of negative a's is even or odd, or if we let p_t be the number of negative a's in any term t the sign of that term will be $(-1)^{p_t}$. The signs of the a's and the b's can be taken in all possible ways, so the number of terms is 2^{n+m} and we can write it as follows:

$$2^{n+m}i^n \prod_{k=1}^{n} \sin a_k \cdot \prod_{j=1}^{m} \cos b_j = \sum_{t=1}^{2^{n+m}} (-1)^{p_t} e^{i(\pm a_1 \pm a_2 \pm \cdots \pm a_n \pm b_1 \pm b_2 \pm \cdots \pm b_m)}. \tag{2}$$

Case 1. *n odd.* For every term e^{iu} there is the term $-e^{-iu}$, for changing the sign of the exponent changes the signs of all the a's, hence changes the number of negative a's from odd to even or from even to odd. But

$$e^{iu} - e^{-iu} = i[\sin u - \sin(-u)];$$

hence we get

$$2^{n+m}i^{n-1} \prod_{k=1}^{n} \sin a_k \cdot \prod_{j=1}^{m} \cos b_j = \sum_{t=1}^{2^{n+m}} (-1)^{p_t} \sin(\pm a_1 \pm a_2 \pm \cdots \pm a_n \pm b_1 \pm \cdots \pm b_m) \tag{3}$$

where the summation means the same as before.

Case 2. *n even.* For every term e^{iu} there is the term e^{-iu} and we have

$$e^{iu} + e^{-iu} = \cos u + \cos(-u);$$

hence we get

$$2^{n+m}i^n \prod_{k=1}^{n} \sin a_k \cdot \prod_{j=1}^{m} \cos b_j = \sum_{t=1}^{2^{n+m}} (-1)^{p_t} \cos(\pm a_1 \pm a_2 \pm \cdots \pm a_n \pm b_1 \pm \cdots \pm b_m). \tag{4}$$

Since $-\sin(-u) = \sin u$ and $\cos(-u) = \cos u$, we have twice as many terms as we need in both (3) and (4). In either (3) or (4), we could take one of the a's (or one of the b's) as always positive and then divide the left member of the equation by two. In (3), we could exclude all the negative terms and then divide the left member of the equation by two. It also should be noticed that in (3) and (4) the power of i is such as to equal either plus or minus one.

As a special case of (3), we have

$$2^n i^{n-1} \prod_{k=1}^{n} \sin a_k = \sum_{t=1}^{2^n} (-1)^{p_t} \sin(\pm a_1 \pm a_2 \pm \cdots \pm a_n)$$

where n is odd. Two special cases of (4) are

$$2^n i^n \prod_{k=1}^{n} \sin a_k = \sum_{t=1}^{2^n} (-1)^{p_t} \cos(\pm a_1 \pm a_2 \pm \cdots \pm a_n)$$

where n is even and

$$2^m \prod_{j=1}^{m} \cos b_j = \Sigma \cos(\pm b_1 \pm b_2 \cdots \pm b_m).$$

The summation in each is to be taken the same as before and as in (3) and (4) contains twice as many terms as necessary. This can be taken care of in the same manner as in (3) and (4).

3046 [1923, 449]. Proposed by A. L. WECHSLER, New York City.

What is the probability that there will be at least r consecutive heads out of n tosses of a coin?

SOLUTION BY OTTO DUNKEL, Washington University.

If $P(n)$ is the number of different ways at least r consecutive heads may turn up in n tosses of a coin, then the probability that at least r consecutive heads will turn up is $p(n) = P(n)/2^n$.

If the coin is tossed $n+1$ times, then, since the $(n+1)$th toss may give a head or tail, $P(n+1)$ is at least as great as $2P(n)$. There are, however, a certain number of unfavorable cases in n tosses which become favorable if the $(n+1)$th toss is a head. These are the cases in which less than r consecutive heads turn up in the first $n-r$ tosses, then a tail, and at last

$r-1$ consecutive heads. The number of such cases is $2^{n-r}-P(n-r)$. We have then for the total number of favorable cases

$$P(n+1)=2P(n)-P(n-r)+2^{n-r}, \tag{1}$$

and then by dividing by 2^{n+1}, we have

$$p(n+1)=p(n)-\frac{p(n-r)}{2^{r+1}}+\frac{1}{2^{r+1}}. \tag{2}$$

This last equation is to be solved with the conditions that $p(n)=0$ when n is less than r, and $p(r)=1/2^r$. If we neglect the term in (2) with the minus sign, then we easily find by setting in (2) $n=r, r+1, \cdots$,

$$p(n)=\frac{n-r+2}{2^{r+1}}, \qquad r\leqq n\leqq 2r. \tag{3}$$

The above is correct for the limited value of n, but in any case we shall have such a term. In the case of n greater than $2r$ we shall have to subtract a function of n and r. By setting in (2) $n=2r, 2r+1, \cdots, 2r+j$, it will easily be seen that this function is of the second degree in j, if $j\leqq r$. Also it must vanish for $j=0$ in order to reduce to (3). Hence we may write the subtracted function as $j(Aj+B)$. Setting this expression in (2) and equating coefficients we find

$$p(2r+j)=\frac{r+j+2}{2^{r+1}}-\frac{1}{2^{2r+2}}\frac{j(j+3)}{2}, \qquad j\leqq r+1. \tag{4}$$

In order to find $p(3r+j)$, it should be observed that the last term will be a polynomial of the third degree in j, which may be seen as before, and it must vanish for $j=0, 1$, in order to reduce to (4). Writing it as $j(j-1)(Aj+B)$ and inserting it in (2), we find on comparing coefficients that

$$p(3r+j)=\frac{2r+j+2}{2^{r+1}}-\left(\frac{1}{2^{r+1}}\right)^2\frac{(r+j)(r+j+3)}{2}+\left(\frac{1}{2^{r+1}}\right)^3\frac{j(j-1)(j+4)}{3!}. \tag{5}$$

It will be seen that this is true for $j\leqq r+2$.

This suggests the form

$$p(ir+j)=\sum_{k=1}^{k=i}(-1)^{k+1}\left(\frac{1}{2^{r+1}}\right)^k {}_{(i-k)r+j}C_{k-1}\frac{(i-k)r+j+k+1}{k}, \qquad j\leqq r+i-1, \tag{6}$$

where ${}_mC_k$ is the coefficient of x^k in the expansion of $(1+x)^m$. It will be found by substitution in (2) that this satisfies the equation for $j\leqq r+i-1$, and hence gives the solution.

For $n=50$, $r=5$, we find $p(50)=.55188$.

3047 [1923, 449]. Proposed by ARNOLD DRESDEN, University of Wisconsin.

Prove that for any positive integer n,

$$\Sigma\frac{1}{\Pi_{i=1}^{t}p_i!\,k_i^{p_i}}=1,$$

where k_i and p_i are positive integers, k_i being the distinct elements of any t-partite partition of $n(t=1\cdots n)$ and p_i the number of parts of the partition which are equal to k_i.

SOLUTION BY E. T. BELL, University of Washington.

The theorem is the result of equating coefficients of x^n ($|x|<1$) in

$$1+x+x^2+\cdots=1+\frac{1}{1!}\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots\right)+\frac{1}{2!}\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots\right)^2+\cdots,$$

which is the expanded form of the identity

$$\frac{1}{1-x}=e^{\log(1/1-x)}.$$

3049 [1923, 450]. Proposed by H. GROSSMAN (Student), College of the City of New York.

Prove that every factor of $2^{2^n}+1$ is congruent to 1 mod 2^{n+1}, and that no two different numbers of the form $2^{2^n}+1$ have a common factor.

SOLUTION BY A. S. WIENER, Cornell University.

Since $2^{2^n}+1$ is odd, its factors must be odd. Let p be any prime factor of $2^{2^n}+1$. Then $2^{2^n}+1\equiv 0$, mod p, and $2^{2^n}\equiv -1$, mod p. Hence $2^{2^{n+1}}\equiv 1$, mod p.

Suppose that the exponent to which 2 belongs, mod p, is d. Then $2^{n+1}=Md$. Hence, $M=2^\nu$, where $\nu\geqq 0$. Then $d=2^{n+1-\nu}$. Suppose $\nu>0$. Then $2^n=2^{\nu-1}2^{n+1-\nu}$. Hence, $2^{2^n}\equiv 1$, mod p, which contradicts $2^{2^n}\equiv -1$, mod p. Our assumption was false and $\nu=0$. Hence, $d=2^{n+1}$, and the exponent to which 2 belongs mod p is 2^{n+1}.

$2^{p-1}\equiv 1$, mod p. Hence, $p-1=M2^{n+1}$ and $p\equiv 1$, mod 2^{n+1}.

Let N be any factor of $2^{2^n}+1$ and assume $N=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_n^{\alpha_n}$. $p_r\equiv 1$, mod 2^{n+1}, by the last congruences. Hence, $p_r^{\alpha_r}\equiv 1$, mod 2^{n+1}, and, therefore, $\prod_1^n p_r^{\alpha_r}\equiv 1$, mod 2^{n+1}. Hence, $N\equiv 1$, mod 2^{n+1}, *i.e.*, every factor of $2^{2^n}+1$ is congruent to 1, mod 2^{n+1}. Suppose $2^{2^n}+1$ and $2^{2^m}+1$ have a common factor N, and let p be a prime factor of N. Then $2^{2^n}+1$ and $2^{2^m}+1$ have a common factor p. Now the exponent of 2 belonging to mod p is 2^{n+1} and 2^{m+1}. This is impossible unless $n=m$. Hence, no two different numbers of the form $2^{2^n}+1$ have a common factor; and the theorem is proved.

A more general problem would be: Prove that every odd factor of $a^{2^n}+1$ is congruent to 1 mod 2^{n+1}; two numbers $a^{2^n}+1$ and $a^{2^m}+1$ have no common odd factor unless $n=m$.

NOTES AND NEWS.

Readers are invited to contribute to the general interest of this department by sending items to R. W. BURGESS, c/o Western Electric Co., 195 Broadway, New York City.

Professor E. C. COKER, for 18 years professor of mathematics at Winthrop College, Rock Hill, S. C., has been appointed professor of astronomy and mathematics at the University of South Carolina. His successor at Winthrop College is Dr. G. T. PUGH.

At Lehigh University, Professor P. A. LAMBERT has been promoted to be head of the department of mathematics and astronomy.

The Franklin Institute of Philadelphia has awarded a Franklin medal and an honorary certificate of membership to Sir ERNEST RUTHERFORD, of Cambridge University.

The National Academy of Sciences has awarded its Watson medal to Professor C. V. L. CHARLIER, of the University of Lund, and its Henry Draper medal to Professor A. S. EDDINGTON, of Cambridge University. Professor Eddington will lecture on general relativity at the University of California during the first semester of 1924–25, and will conduct a seminar on sidereal astronomy.

New York University has conferred the honorary degree of doctor of laws on Professor M. I. PUPIN, of Columbia University.

Professor A. B. COBLE has been elected a member of the National Academy of Sciences.

Dr. B. H. BROWN has been promoted to an assistant professorship of mathematics at Dartmouth College.

Assistant Professor HARRIS RICE, of Worcester Polytechnic Institute, has been promoted to a full professorship.

Dr. NORBERT WIENER, of the Massachusetts Institute of Technology, has been promoted to an assistant professorship of mathematics.

Assistant Professor ERNEST FLAMMER, of Queen's University, Kingston, Ontario, has been promoted to an associate professorship of mathematics.

Assistant Professor E. J. OGLESBY, of New York University, has been promoted to an associate professorship of mathematics.

Assistant Professor H. S. EVERETT, of Bucknell University, has been promoted to an associate professorship of mathematics.

Dr. J. A. NYSWANDER has been appointed assistant professor of mathematics at Swarthmore College.

Associate Professor J. J. LUCK, of the University of Virginia, has been promoted to a full professorship of mathematics.

Dr. L. T. E. THOMPSON has been appointed physicist at the Naval Proving Ground, Dahlgren, Va.

Assistant Professor R. A. SHEETS, of Denison University, has been appointed assistant professor of mathematics at Miami University.

Assistant Professor W. G. SIMON, of Western Reserve University, has been promoted to an associate professorship of mathematics.

Assistant Professor C. O. WILLIAMSON, of the College of Wooster, has been promoted to a full professorship of mathematics.

Assistant Professor F. W. REED, of Ohio University, has been promoted to an associate professorship.

Assistant Professor J. J. NASSAU, of the Case School of Applied Science, has been promoted to an associate professorship of mathematics and astronomy.

Associate Professor C. L. ARNOLD, of Ohio State University, has been promoted to a full professorship of mathematics.

At the University of Cincinnati, Associate Professor C. N. MOORE has been promoted to a full professorship and Dr. I. A. BARNETT to an assistant professorship of mathematics.

Professor W. A. HAMILTON, formerly of Beloit College, who has been a special lecturer at the University of Wisconsin during the past year, has been appointed head of the department of mathematics at Antioch College, Yellow Springs, Ohio.

Associate Professor K. P. WILLIAMS, of Indiana University, has been promoted to a full professorship of mathematics.

Assistant Professor L. C. EMMONS, of Michigan Agricultural College, has been promoted to a full professorship of mathematics.

At the University of Michigan, Associate Professors J. W. BRADSHAW and T. H. HILDEBRANDT have been promoted to full professorships. Assistant Professor C. J. COE and Instructors W. M. COATES and D. K. KAZARINOFF are on leave of absence studying in Europe.

Dr. C. A. GARABEDIAN, of Harvard University, has been appointed assistant professor of mathematics at Northwestern University.

President W. A. GRANVILLE, of Gettysburg College, has resigned to become educational director of the United States National Life and Casualty Company, Chicago.

Associate Professor W. D. MACMILLAN, of the University of Chicago, has been promoted to a full professorship of mathematical astronomy.

Dr. FREDRICK WOOD, of the University of Wisconsin, has been appointed professor of mathematics at Lake Forest College.

At the University of Illinois, Dr. C. C. CAMP has been promoted to be an associate in mathematics. Professor H. BLUMBERG has been granted leave of absence for the academic year 1924–1925.

At the University of Missouri, Professor G. E. WAHLIN, of the University of Illinois, has been appointed professor; Associate Professor LOUIS INGOLD has been promoted to a full professorship, to become effective at the expiration of his leave of absence for the year 1924–1925; Dr. E. F. ALLEN has been promoted to an assistant professorship; and Dr. HERMAN BETZ, of Yale University, has been appointed assistant professor.

Mr. L. V. ROBINSON, of the University of Virginia, has been appointed assistant professor of mathematics and astronomy at Oklahoma City College.

Mr. W. M. WHYBURN, of the South Park Junior College, Beaumont, Texas, has been appointed assistant professor of mathematics at Texas Agricultural and Mechanical College.

Miss MARY CAMPBELL, of the University of Texas, has been appointed professor and head of the department of mathematics at South Park Junior College.

Associate Professor G. H. CRESSE, of the University of Arizona, has been promoted to a full professorship of mathematics.

Associate Professor I. L. MILLER, of South Dakota State College, has been promoted to a full professorship of mathematics.

Dr. F. M. WEIDA, of Iowa State University, has been appointed assistant professor of mathematics at Montana State College.

Assistant Professor L. S. DEDERICK, of the United States Naval Academy, has been appointed professor of mathematics at the University of British Columbia.

At the University of California, Assistant Professors B. A. BERNSTEIN and FRANK IRWIN have been promoted to associate professorships and Dr. SOPHIA H. LEVY has been promoted to an assistant professorship of mathematics. At the Southern Branch, at Los Angeles, Associate Professor G. E. F. SHERWOOD

has been promoted to a full professorship, and Dr. P. H. DAUS to an assistant professorship.

Dr. HAROLD HOTELLING, of Princeton University, has been appointed junior assistant at the Food Research Institute, Stanford University.

Dr. H. M. JEFFERS, of Iowa State University, has been appointed assistant astronomer at the Lick Observatory.

Dr. JAMES OUSPENSKY of Leningrad, a delegate from the Russian Academy of Sciences to the meeting of the International Mathematical Congress at Toronto, was a recent visitor at the University of Michigan. While at Michigan, he presented to the mathematics club a summary of some remarkable results obtained in 1917 by a student of his, Vinogradoff by name, on the distribution of residues and non-residues of powers.

Associate Professor R. W. BRINK is in Europe on sabbatical leave from the University of Minnesota.

The following appointments to instructorships are announced:

Wellesley College, Miss ETHEL L. ANDERTON;
Smith College, Miss BESS M. EVERSULL;
New York University, Dr. CONSTANCE R. BALLANTINE;
Lehigh University, Mr. C. A. BALOF;
University of Florida, Mr. C. G. PHIPPS;
Purdue University, Mr. J. C. BENNETT;
University of Iowa (astronomy), Dr. D. H. MENZEL;
University of Missouri, Dr. L. H. MACFARLAN;
University of Wisconsin, Mr. H. S. POLLARD;
University of Michigan, Dr. J. A. SHOHAT, Messrs. A. P. MASLOW, G. S. VAN FLEET, W. C. GREEN.

Professor J. E. HODGSON, of West Virginia University, died April 11, 1924.

Dr. R. S. WOODWARD, retired president of the Carnegie Institution, died June 30, 1924, at the age of seventy-four years.

Immediately after the close of the International Mathematical Congress at Toronto, Professor CHARLES DE LA VALLÉE POUSSIN of the University of Louvain started upon a lecture tour in the United States under the auspices of the Educational Foundation of the Commission for Belgian Relief. His first visit was at the University of Chicago where he stayed for a week, during which time he gave three lectures in French on the general topic "L'approximation des fonctions de variables réelles et les fonctions quasi-analytiques." These lectures were commented upon in English, topic by topic, by Professor MAURICE FRÉCHET of the University of Strasbourg, who was a member of the Chicago staff for the summer quarter. These lectures were greatly appreciated by the members of the Chicago faculty and the two hundred or more graduate students in the department. Similar lectures were subsequently given by Professor DE LA VALLÉE POUSSIN at the Universities of California, Michigan, Minnesota, Wisconsin and other universities.

Work has been slowly going on for some years past at the Astrophysical Observatory of Arcetri, near Florence, Italy, on the hill sacred to the memory of Galileo (for it was here that he made many of his observations and discoveries) for the building of a Sun-Tower similar to that already existing near the Mount Wilson Observatory in California. The Tower is very near Galileo's last residence, which was also the scene of his death, and is intended to serve as a lasting memorial to this great master of science of the sixteenth century. Had it not been for the unexpected increase in the price of materials due to the war, the Tower would have been finished some years ago. Owing however to this cause, it is estimated that, notwithstanding a recent generous contribution from the Ministry of Public Instruction of Italy, a further sum of 30,000 Lire will be needed for its completion. Contributions are therefore requested from scientists and others in all parts of the world. They should be addressed to Cav. Isacco Ciabattari, *Società Leonardo da Vinci, Via dei Corsi, 5, Firenze, Italy.*

The following research courses in mathematics are to be offered in 1924–25 at the "Institute De Mathématiques" de l'Universite de Strasbourg, France, in addition to the usual fundamental courses on analysis, astronomy and mechanics. These research courses are particularly fitted to prepare candidates for the "Diplôme d'études superieures de mathématiques" and for the Doctorates.

First Semester (Nov. 1924 to Febr. 1925). By Professor BAUER, Theory of quanta (2 hours).—By Professor CERF, Singular solutions of differential equations (1 hour).—By Professor FRÉCHET, Theory of abstract sets (3 hours).

Second Semester (March to June, 1925). By Professor BAUER, Constitution of atoms (2 hours).—By Professor FRÉCHET, Smoothing of empirical functions (3 hours).—By Professor THIRY, Hydrodynamics (Selected topics) (2 hours).—By Professor VALIRON, A new theory of integral and meromorphic functions (2 hours).—By Professor VILLAT, On some generalizations of Lamé's differential equations and on minimal surfaces (2 hours).

Isis, an International Review devoted to the History of Science and Civilization, is the official organ of the History of Science Society. It is edited by GEORGE SARTON with associate editors: C. H. HASKINS of Harvard University, R. C. ARCHIBALD of Brown University, J. K. WRIGHT of the American Geographical Society. F. BARRY of Columbia University edits its "Queries and Answers," H. E. BARNES of Smith College its "Teaching and Personalia," and F. E. BRASCH of the Dept. of Terrestrial Magnetism, Washington, D. C., its "News of the History of Science Society" and "Obituaries."

A new edition of the final report of the National Committee on Mathematical Requirements, entitled *The Reorganization of Mathematics in Secondary Education,* has been printed and is ready for distribution. A nominal charge of twenty cents per copy has been found necessary in order to defray packing and shipping costs. Orders for this report with the necessary remittance should be sent to the Dartmouth Press, Hanover, New Hampshire.

THE CARUS MONOGRAPHS.

The first Carus Monograph is ready for the printer. It is entitled: "*The Calculus of Variations*" by G. A. BLISS, of the University of Chicago, and is intended for readers who have not specialized in mathematics beyond the calculus. It will make a book of about 190 pages, size of type page 3½ by 6 inches, and will contain 45 wax engravings. It will be printed in clear readable type, with ample margins, on high quality paper, and will be bound in stiff cloth covers.

It will be published for the Mathematical Association of America by the Open Court Publishing Company, of Chicago, Illinois, and will be sold by the Association directly to its individual and institutional members at a pre-publication price of $1.25 per copy, covering the actual cost of production and mailing in separate cartons. It will be sold after publication to the general public at $2.00 per copy by the Open Court Publishing Company with its widely distributed clientele both in America and in foreign countries.

The Committee of Publication feel justified in assuming that every member of the Association will desire to subscribe for this and the subsequent numbers of the Carus Monograph series, publication of which is made possible by the generous gift of MRS. MARY HEGELER CARUS, sole trustee of the Edward C. Hegeler Trust Fund. For a full account of the terms and spirit of this gift see this MONTHLY, October, 1921, pages 352–354, and June, 1923, pages 151–155. In order to simplify the procedure as much as possible for both officers and members of the Association, the Secretary will merely add this item of $1.25 to the statement of annual dues for 1925 to be sent out in December, 1924, and a copy of Monograph number one will be mailed to each member as soon as his return shall have been received,—unless this item is crossed off.

The manuscript of the second Monograph is also nearly ready on "*Functions of a Complex Variable*" by D. R. CURTISS of Northwestern University. Its publication will follow as soon as practicable.

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DEC 23 1924

THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

EDITED BY

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HERBERT ELLSWORTH SLAUGHT

AUDREY JOHN KEMPNER

WITH THE COÖPERATION OF

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THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XXXI, 1924

NUMBER 9, NOVEMBER

PUBLISHED BY THE ASSOCIATION

LANCASTER, PA., AND ANN ARBOR, MICH.

Entered at the Post Office at Lancaster, Pa., as Second Class Matter

PUBLISHED MONTHLY

$4.00 a Year, Single Copies 45 cents, to Members

$5.00 a Year, Single Copies 60 cents, to Others